PART B

# Unit - IV

# *Curve Fitting and Optimization*

## 4.1 Introduction

In this unit we first discuss the topic ***curve fitting***. This is a method of finding a specific relation connecting the dependent and independent variables for a given data so as to satisfy the data as accurately as possible. Such a curve is called the *curve of best fit*.

***Optimization*** is a technique of obtaining the best results under the given/prevailing circumstances. The technique is used for utilization of resources in industry, business management, economy planning, agriculture etc.

## 4.2 Curve fitting by least squares method

Curve fitting is a method of finding a suitable relation or law in the form $y = f(x)$ for a set of observed values $(x_i, y_i), i = 1, 2, \ldots n$.

Such a relation connecting $x$ and $y$ is known as *empirical law*.

This relation is most suitable for predicting/estimating the value of $y$ for a given value of $x$. The method of least squares is as follows.

Suppose $y = f(x)$ is an approximate relation that fits into a given data comprising $(x_i, y_i), i = 1, 2, \ldots n$ then $y_i$'s are called observed values and $Y_i = f(x_i)$ are called expected values. Their difference $R_i = y_i - Y_i$ are called the residuals or estimate errors.

The *method of least squares* provides a relationship $y = f(x)$ such that the sum of the squares of the residuals is least.

### 4.21 Fitting of a straight line $y = ax + b$

Consider a set of $n$ given values $(x, y)$ for fitting the straight line $y = ax + b$ where $a$ and $b$ are parameters to be determined. The residual $R = y - (ax + b)$ is the difference between the observed and estimated values of $y$. By the method of least squares we find parameters $a$ and $b$ such that the sum of squares of the residuals is minimum (*least*).

Let $\quad S = \sum_{1}^{n} R^2$

i.e., $\quad S = \sum_{1}^{n} [y - (ax+b)]^2$

Treating $S$ as a function of two parameters $a$ and $b$ the necessary conditions for $S$ to be minimum are $\frac{\partial S}{\partial a} = 0$ and $\frac{\partial S}{\partial b} = 0$

i.e., $\quad 2\sum_{1}^{n} [y - (ax+b)](-x) = 0$

and $\quad 2\sum_{1}^{n} [y - (ax+b)](-1) = 0$

Dividing both the equations by 2 we have

$$-\sum_{1}^{n} xy + \sum_{1}^{n} a x^2 + \sum_{1}^{n} b x = 0$$

$$-\sum_{1}^{n} y + \sum_{1}^{n} a x + \sum_{1}^{n} b = 0$$

But $\quad \sum_{1}^{n} b = b + b + b + \ldots\ n$ times $= nb$ and hence we have

$$a\sum x^2 + b\sum x = \sum xy$$

$$a\sum x + nb = \sum y$$

These equations are called ***normal equations*** for fitting the straight line $y = ax + b$ in the least squares sense. By solving these we obtain the value of $a$ and $b$.

## 4.22 Fitting of a second degree parabola $y = ax^2 + bx + c$

Consider a set of $n$ given values $(x, y)$ for fitting the curve $y = ax^2 + bx + c$.

The residual $R = y - (ax^2 + bx + c)$ is the difference between the observed and estimated values of $y$. We have to find parameters $a, b, c$ such that the sum of the squares of the residuals is the least.

Let $\quad S = \sum_{1}^{n} [y - (ax^2 + bx + c)]^2$

Treating $S$ as a function of three parameters $a, b, c$ the necessary conditions for $S$ to be minimum are $\frac{\partial S}{\partial a} = 0$ , $\frac{\partial S}{\partial b} = 0$ , $\frac{\partial S}{\partial c} = 0$

i.e., $$2\sum_{1}^{n}[y-(ax^2+bx+c)](-x^2) = 0$$

$$2\sum_{1}^{n}[y-(ax^2+bx+c)](-x) = 0$$

$$2\sum_{1}^{n}[y-(ax^2+bx+c)](-1) = 0$$

Dividing all these equations by 2 we have,

$$-\sum_{1}^{n}x^2y+\sum_{1}^{n}ax^4+\sum_{1}^{n}bx^3+\sum_{1}^{n}cx^2 = 0$$

$$-\sum_{1}^{n}xy+\sum_{1}^{n}ax^3+\sum_{1}^{n}bx^2+\sum_{1}^{n}cx = 0$$

$$-\sum_{1}^{n}y+\sum_{1}^{n}ax^2+\sum_{1}^{n}bx+\sum_{1}^{n}c = 0$$

But $\sum_{1}^{n}c = c+c+c+\ldots n$ times $= nc$ and hence we have

$$a\sum x^4+b\sum x^3+c\sum x^2 = \sum x^2y$$

$$a\sum x^3+b\sum x^2+c\sum x = \sum xy$$

$$a\sum x^2+b\sum x+nc = \sum y$$

These equations are called ***normal equations*** for fitting the second degree parabola $y = ax^2+bx+c$ in the least square sense. By solving these we obtain the value of $a, b, c$.

**Note** : *The normal equations for fitting a straight line or parabola can be written instantly from the desired equation of the curve as follows.*

*We first apply summation ( $\sum$ ) to the desired equation keeping the constants $a, b, c$ outside the summation where the summation of pure constant terms like $\sum a$, $\sum b$, $\sum c$ are to be written as $na$, $nb$, $nc\ldots$ respectively.*

*We then multiply the given equation by the independent variable $x$ and apply summation again. This will suffice for fitting a straight line. However in the case of parabola we must also multiply by $x^2$ and apply summation.*

## 4.23 Fitting of a curve of the form $y = ae^{bx}$

Consider $y = ae^{bx}$. Taking logarithms *(to the base e)* on both sides we get

$\log_e y = \log_e a + bx\log_e e$. But $\log_e e = 1$

ie., $\log_e y = \log_e a + bx$

or $Y = A+BX$ . . . (1)

where $Y = \log_e y$, $A = \log_e a$, $B = b$, $X = x$

It is evident that (1) is the equation of a straight line and the associated *normal equations* are as follows.

$$\sum Y = nA + B\sum X \qquad \ldots (2)$$

$$\sum XY = A\sum X + B\sum X^2 \qquad \ldots (3)$$

Solving (2) and (3) we obtain $A$ and $B$. But $\log_e a = A \Rightarrow a = e^A$ ; Also $b = B$. Substituting these values in $y = a\,e^{bx}$ we get the curve of best fit, in the required form.

## 4.24 Fitting of a curve of the form $y = a\,x^b$

Consider $y = a\,x^b$

Taking logarithms *(to the base e)* on both sides we get,

$$\log_e y = \log_e a + b\log_e x$$

or $\quad Y = A + BX \qquad \ldots (1)$

where $Y = \log_e y,\ \ A = \log_e a,\ \ B = b,\ \ X = \log_e x$

The normal equations associated with (1) are same as in the previous case and we can obtain $a$, $b$. Substituting $a$ , $b$ in $y = a\,x^b$ we get the curve of best fit in the required form.

### Remarks

*The curves $y = a\,e^{bx}$, $y = a\,x^b$ are called exponential curves and these equations are obtained by modifying the same into the form of a straight line. $y = a\,b^x$ also admits the same procedure. A few more equations for the benefit of readers are presented in the following table which can be reduced to the form of a straight line by suitable modification and substitution.*

| | Desired equation | Modification | Substitution | Reduced Form |
|---|---|---|---|---|
| 1. | $y = ax^n + b$ | - | $X = x^n$ | $y = aX + b$ |
| 2. | $y = a + \frac{b}{x}$ | - | $X = \frac{1}{x}$ | $y = a + bX$ |
| 3. | $y = ax + \frac{b}{x}$ | Multipy by $x$ to obtain $xy = ax^2 + b$ | $Y = xy,\ X = x^2$ | $Y = aX + b$ |
| 4. | $y = \frac{x}{a + bx}$ | $\frac{1}{y} = \frac{a + bx}{x}$ or $\frac{1}{y} = \frac{a}{x} + b$ | $Y = \frac{1}{y},\ X = \frac{1}{x}$ | $Y = aX + b$ |
| 5. | $xy^n = a$ | $\log_e x + n\log_e y = \log_e a$ or $\log_e y = \frac{1}{n}\left[\log_e a - \log_e x\right]$ | $Y = \log_e y$, $A = -\frac{1}{n}$, $B = \frac{1}{n}\log_e a$ | $Y = AX + B$ |

**Working procedure for problems**

- We first write the normal equations approriate to the curve of best fit.
- We prepare the relevant table and find the value of all the summations present in the normal equations.
- We substitute these values to arrive at a system of equations in the unknown parameters.
- We find the parameters by solving the system of equations and substitute in the equation $y = f(x)$ which being the curve of best fit.

**Note** : **1.** *The solution of the system of equations in 2 or 3 unknown can be obtined directly from the latest commonly used scientific calculators.* [ *Version : Casio* $fx - 100$ , $fx - 991\ MS$ , $fx - 991\ ES$ ]

**2.** *If the magnitudes involved in the data are big, we may modify the data choosing a convenient origin somewhere in the middle and finally revert back* [ *See problem : 5* ]

## WORKED PROBLEMS

1. *Fit a straight line* $y = ax + b$ *for the following data.*

| $x$ | 1 | 3 | 4 | 6 | 8 | 9 | 11 | 14 |
|---|---|---|---|---|---|---|---|---|
| $y$ | 1 | 2 | 4 | 4 | 5 | 7 | 8 | 9 |

>> The normal equations for fitting the straight line $y = ax + b$ are

$$\sum y = a \sum x + nb$$

$$\sum xy = a \sum x^2 + b \sum x \qquad (n = 8)$$

The relevant table is as follows.

| $x$ | $y$ | $xy$ | $x^2$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 3 | 2 | 6 | 9 |
| 4 | 4 | 16 | 16 |
| 6 | 4 | 24 | 36 |
| 8 | 5 | 40 | 64 |
| 9 | 7 | 63 | 81 |
| 11 | 8 | 88 | 121 |
| 14 | 9 | 126 | 196 |
| $\sum x = 56$ | $\sum y = 40$ | $\sum xy = 364$ | $\sum x^2 = 524$ |

The normal equations become

$$56a + 8b = 40$$

$$524a + 56b = 364$$

On solving ( *Using calculator* ) we have,

$$a = 0.636363636 \approx 0.64 \quad ; \quad b = 0.545454545 \approx 0.55$$

Thus by substituting these values in $y = ax + b$ we obtain the equation,

$$\mathbf{y = 0.64\,x + 0.55}$$

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**2.** *Find the equation of the best fitting straight line for the following data and hence estimate the value of the dependent variable corresponding to the value 30 of the independent variable.*

| $x$ | 5 | 10 | 15 | 20 | 25 |
|---|---|---|---|---|---|
| $y$ | 16 | 19 | 23 | 26 | 30 |

>> Let $y = ax + b$ be the equation of the best fitting straight line. The associated normal equations are as follows.

$$\sum y = a \sum x + nb$$

$$\sum xy = a \sum x^2 + b \sum x \quad (n = 5)$$

The relevant table is as follows.

| $x$ | $y$ | $xy$ | $x^2$ |
|---|---|---|---|
| 5 | 16 | 80 | 25 |
| 10 | 19 | 190 | 100 |
| 15 | 23 | 345 | 225 |
| 20 | 26 | 520 | 400 |
| 25 | 30 | 750 | 625 |
| $\sum x = 75$ | $\sum y = 114$ | $\sum xy = 1885$ | $\sum x^2 = 1375$ |

The normal equations become

$$75a + 5b = 114$$

$$1375a + 75b = 1885$$

On solving ( *Using calculator* ) we have, $a = 0.7$, $b = 12.3$

Thus by substituting these values in $y = ax + b$ we obtain the equation of the best fitting straight line in the form

$$y = 0.7x + 12.3$$

Further **when** $x = 30$ we obtain $y = 0.7(30) + 12.3 = \mathbf{33.3}$

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3. *Fit a straight line in the least square sense for the following data*

| | | | | |
|---|---|---|---|---|
| $x$ | 50 | 70 | 100 | 120 |
| $y$ | 12 | 15 | 21 | 25 |

>> Let $y = ax + b$ be the equation of the best fitting straight line in the least square sense. The associated normal equations are as follows.

$$\sum y = a \sum x + n\, b$$

$$\sum xy = a \sum x^2 + b \sum x \quad (n = 4)$$

The relevant table is as follows.

| $x$ | $y$ | $xy$ | $x^2$ |
|---|---|---|---|
| 50 | 12 | 600 | 2500 |
| 70 | 15 | 1050 | 4900 |
| 100 | 21 | 2100 | 10000 |
| 120 | 25 | 3000 | 14400 |
| $\sum x = 340$ | $\sum y = 73$ | $\sum xy = 6750$ | $\sum x^2 = 31800$ |

The normal equations become

$$340\,a + 4\,b = 73$$

$$31800\,a + 340\,b = 6750$$

On solving ( *Using calculator* ) we have

$$a = 0.187931034 \approx 0.19,\; b = 2.275862069 \approx 2.28$$

Thus the required equation of the straight line in the least square sense is

$$\mathbf{y = 0.19\,x + 2.28}$$

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4. *A simply supported beam carries a concentrated load P at its mid point. Corresponding to various values of P the maximum deflection Y is measured and is given in the following table.*

| P | 100 | 120 | 140 | 160 | 180 | 200 |
|---|---|---|---|---|---|---|
| Y | 0.45 | 0.55 | 0.60 | 0.70 | 0.80 | 0.85 |

*Find a law of the form $Y = a + bP$ and hence estimate Y when P is 150.*

>> The normal equations associated with $Y = a + bP$ are as follows.

$$\sum Y = na + b \sum P$$

$$\sum PY = a \sum P + b \sum P^2 \quad (n = 6)$$

The relevant table is as follows.

| $P$ | $Y$ | $PY$ | $P^2$ |
|---|---|---|---|
| 100 | 0.45 | 45 | 10000 |
| 120 | 0.55 | 66 | 14400 |
| 140 | 0.60 | 84 | 19600 |
| 160 | 0.70 | 112 | 25600 |
| 180 | 0.80 | 144 | 32400 |
| 200 | 0.85 | 170 | 40000 |
| $\sum P = 900$ | $\sum Y = 3.95$ | $\sum PY = 621$ | $\sum P^2 = 142000$ |

The normal equations become

$$6a + 900b = 3.95$$

$$900a + 142000b = 621$$

On solving ( *Using calculator* ) we have

$$a = 0.047619047 \approx 0.0476 \;;\; b = 0.0040701429 \approx 0.0041$$

Thus the required law is $Y = 0.0476 + 0.0041\,P$.

Also **when** $P = 150$, $Y = 0.6626 \approx 0.66$

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5. *Fit a straight line to the following data.*

| Year | 1961 | 1971 | 1981 | 1991 | 2001 |
|---|---|---|---|---|---|
| Production (in tons) | 8 | 10 | 12 | 10 | 16 |

*Also find the expected production in the year 2006.*

>> Let year and production respectively be represented by the variables $x$ and $y$. We shall fit the straight line in the form $y = a + bx$. Since the values of $x$ are large in magnitude, we *may* prefer to modify the same by choosing a convenient origin somewhere in the middle.

Let $X = x - 1981$ and the line of fit will be $y = a + bX$

[**Note** : *If the data was upto* 1991 *(say) we can take* $X = x - 1975$]

The normal equations associated with $y = a + bX$ are as follows.

$$\sum y = na + b\sum X$$

$$\sum Xy = a\sum X + b\sum X^2 \quad (n = 5)$$

The relevant table is as follows

| $X$ | $y$ | $Xy$ | $X^2$ |
|---|---|---|---|
| −20 | 8 | −160 | 400 |
| −10 | 10 | −100 | 100 |
| 0 | 12 | 0 | 0 |
| 10 | 10 | 100 | 100 |
| 20 | 16 | 320 | 400 |
| $\sum X = 0$ | $\sum y = 56$ | $\sum Xy = 160$ | $\sum X^2 = 1000$ |

The normal equations become,

$$5a = 56 \quad \text{and} \quad 1000b = 160$$

$$\therefore \quad a = 11.2 \quad \text{and} \quad b = 0.16$$

Hence $y = a + bX$, with $X = x - 1981$ becomes

$$y = 11.2 + 0.16\,(x - 1981)$$

Thus $\mathbf{y = -305.76 + 0.16\,x}$ is the required line of fit.

Also when $x = 2006$, $y = -305.76 + 0.16\,(2006) = 15.2$

**Expected production in the year 2006 is 15.2 tons**

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6. *Fit a second degree parabola* $y = ax^2 + bx + c$ *in the least square sense for the following data and hence estimate* $y$ *at* $x = 6$.

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $y$ | 10 | 12 | 13 | 16 | 19 |

>> The normal equations associated with $y = a x^2 + b x + c$ are as follows.

$$\sum y = a \sum x^2 + b \sum x + n c$$

$$\sum xy = a \sum x^3 + b \sum x^2 + c \sum x$$

$$\sum x^2 y = a \sum x^4 + b \sum x^3 + c \sum x^2 \quad (n = 5)$$

The relevant table is as follows.

| $x$ | $y$ | $xy$ | $x^2 y$ | $x^2$ | $x^3$ | $x^4$ |
|---|---|---|---|---|---|---|
| 1 | 10 | 10 | 10 | 1 | 1 | 1 |
| 2 | 12 | 24 | 48 | 4 | 8 | 16 |
| 3 | 13 | 39 | 117 | 9 | 27 | 81 |
| 4 | 16 | 64 | 256 | 16 | 64 | 256 |
| 5 | 19 | 95 | 475 | 25 | 125 | 625 |
| $\sum x = 15$ | $\sum y = 70$ | $\sum xy = 232$ | $\sum x^2 y = 906$ | $\sum x^2 = 55$ | $\sum x^3 = 225$ | $\sum x^4 = 979$ |

The normal equations become

$$55a + 15b + 5c = 70$$

$$225a + 55b + 15c = 232$$

$$979a + 225b + 55c = 906$$

On solving (*Using calculator*) we have

$$a = 0.2857 \approx 0.29,\ b = 0.4857 \approx 0.49,\ c = 9.4.$$

Thus the required second degree parabola is

$\mathbf{y = 0.29\,x^2 + 0.49\,x + 9.4}$ **Also at** $\mathbf{x = 6,\ y = 22.78}$

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7. *Fit a parabola* $y = a + b x + c x^2$ *for the data*

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $y$ | 1 | 1.8 | 1.3 | 2.5 | 2.3 |

>> The normal equations associated with $y = a + b x + c x^2$ are as follows.

$$\sum y = n a + b \sum x + c \sum x^2$$

$$\sum xy = a \sum x + b \sum x^2 + c \sum x^3$$

$$\sum x^2 y = a \sum x^2 + b \sum x^3 + c \sum x^4 \quad (n = 5)$$

The relevant table is as follows.

| $x$ | $y$ | $xy$ | $x^2y$ | $x^2$ | $x^3$ | $x^4$ |
|---|---|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1.8 | 1.8 | 1.8 | 1 | 1 | 1 |
| 2 | 1.3 | 2.6 | 5.2 | 4 | 8 | 16 |
| 3 | 2.5 | 7.5 | 22.5 | 9 | 27 | 81 |
| 4 | 2.3 | 9.2 | 36.8 | 16 | 64 | 256 |
| $\sum x = 10$ | $\sum y = 8.9$ | $\sum xy = 21.1$ | $\sum x^2y = 66.3$ | $\sum x^2 = 30$ | $\sum x^3 = 100$ | $\sum x^4 = 354$ |

The normal equations become

$$5a + 10b + 30c = 8.9$$

$$10a + 30b + 100c = 21.1$$

$$30a + 100b + 354c = 66.3$$

On solving ( *Using calculator* ) we have, $a = 1.0771$, $b = 0.4157$, $c = -0.0214$

Thus the parabola of fit is

$$\mathbf{y = 1.0771 + 0.4157x - 0.0214x^2}$$

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8. *Fit a parabola $y = ax^2 + bx + c$ by the method of least squares for the following data.*

| $x$ | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|
| $y$ | 3.07 | 12.85 | 31.47 | 57.38 | 91.29 |

>> The normal equations assocated with $y = ax^2 + bx + c$ are as follows.

$$\sum y = a\sum x^2 + b\sum x + nc$$

$$\sum xy = a\sum x^3 + b\sum x^2 + c\sum x$$

$$\sum x^2 y = a\sum x^4 + b\sum x^3 + c\sum x^2 \quad (n = 5)$$

The relevant table is as follows.

| $x$ | $y$ | $xy$ | $x^2 y$ | $x^2$ | $x^3$ | $x^4$ |
|---|---|---|---|---|---|---|
| 2 | 3.07 | 6.14 | 12.28 | 4 | 8 | 16 |
| 4 | 12.85 | 51.40 | 205.60 | 16 | 64 | 256 |
| 6 | 31.47 | 188.82 | 1132.92 | 36 | 216 | 1296 |
| 8 | 57.38 | 459.04 | 3672.32 | 64 | 512 | 4096 |
| 10 | 91.29 | 912.90 | 9129.00 | 100 | 1000 | 10000 |
| $\sum x$ **= 30** | $\sum y$ **= 196.06** | $\sum xy$ **= 1618.3** | $\sum x^2 y$ **= 14152.12** | $\sum x^2$ **= 220** | $\sum x^3$ **= 1800** | $\sum x^4$ **= 15664** |

The normal equations become

$$220\,a + 30\,b + 5\,c = 196.06$$

$$1800\,a + 220\,b + 30\,c = 1618.3$$

$$15664\,a + 1800\,b + 220\,c = 14152.12$$

On solving ( *Using calculator* ) we have,

$$a = 0.99196\,, \quad b = -0.85507\,, \quad c = 0.696$$

Thus the required parabola of fit is

$$\mathbf{y = 0.992\,x^2 - 0.855x + 0.696}$$

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9. *Fit a parabola for the following data.*

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $y$ | 2 | 6 | 7 | 8 | 10 | 11 | 11 | 10 | 9 |

>> Let $y = a + bx + cx^2$ be the parabola of fit. The associated normal equations are

$$\sum y = n\,a + b\sum x + c\sum x^2$$

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3$$

$$\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4 \qquad (n = 9)$$

The relevant table is as follows.

| $x$ | $y$ | $xy$ | $x^2 y$ | $x^2$ | $x^3$ | $x^4$ |
|---|---|---|---|---|---|---|
| 1 | 2 | 2 | 2 | 1 | 1 | 1 |
| 2 | 6 | 12 | 24 | 4 | 8 | 16 |
| 3 | 7 | 21 | 63 | 9 | 27 | 81 |
| 4 | 8 | 32 | 128 | 16 | 64 | 256 |
| 5 | 10 | 50 | 250 | 25 | 125 | 625 |
| 6 | 11 | 66 | 396 | 36 | 216 | 1296 |
| 7 | 11 | 77 | 539 | 49 | 343 | 2401 |
| 8 | 10 | 80 | 640 | 64 | 512 | 4096 |
| 9 | 9 | 81 | 729 | 81 | 729 | 6561 |
| $\sum x = 45$ | $\sum y = 74$ | $\sum xy = 421$ | $\sum x^2 y = 2771$ | $\sum x^2 = 285$ | $\sum x^3 = 2025$ | $\sum x^4 = 15333$ |

The normal equations become

$$9a + 45b + 285c = 74$$

$$45a + 285b + 2025c = 421$$

$$285a + 2025b + 15333c = 2771$$

On solving *( Using calculator )* we have,

$$a = -0.92857, \quad b = 3.52316, \quad c = -0.26731$$

Thus the required parabola of fit is

$$\mathbf{y = -0.93 + 3.52x - 0.27x^2}$$

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**10.** *Find the best values of $a, b, c$ if the equation $y = a + bx + cx^2$ is to fit most closely to the following observations.*

| $x$ | $-2$ | $-1$ | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $y$ | $-3.150$ | $-1.390$ | 0.620 | 2.880 | 5.378 |

\>> The normal equations associated with $y = a + bx + cx^2$ are as follows.

$$\sum y = na + b\sum x + c\sum x^2$$

$$\sum xy = a\sum x + b\sum x^2 + c\sum x^3$$

$$\sum x^2 y = a\sum x^2 + b\sum x^3 + c\sum x^4 \quad (n = 5)$$

The relevant table is as follows.

| $x$ | $y$ | $xy$ | $x^2 y$ | $x^2$ | $x^3$ | $x^4$ |
|---|---|---|---|---|---|---|
| $-2$ | $-3.15$ | 6.30 | $-12.60$ | 4 | $-8$ | 16 |
| $-1$ | $-1.39$ | 1.39 | $-1.39$ | 1 | $-1$ | 1 |
| 0 | 0.62 | 0 | 0 | 0 | 0 | 0 |
| 1 | 2.88 | 2.88 | 2.88 | 1 | 1 | 1 |
| 2 | 5.378 | 10.756 | 21.512 | 4 | 8 | 16 |
| $\sum x = 0$ | $\sum y = 4.338$ | $\sum xy = 21.326$ | $\sum x^2 y = 10.402$ | $\sum x^2 = 10$ | $\sum x^3 = 0$ | $\sum x^4 = 34$ |

The normal equations become

$$5a + 10c = 4.338$$

$$10b = 21.326 \quad \therefore \quad b = 2.1326$$

$$10a + 34c = 10.402 \qquad \ldots (2)$$

On solving (1) and (2) [ *Using calculator* ] we have, $a = 0.6210, c = 0.1233$

Thus the best values $a$, $b$, $c$ for fitting the parabola $y = a + bx + cx^2$ are

$$\mathbf{a = 0.621, \quad b = 2.1326, \quad c = 0.1233}$$

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**11.** *Fit a curve of the form* $y = a e^{bx}$ *for the data*

| $x$ | 0 | 2 | 4 |
|---|---|---|---|
| $y$ | 8.12 | 10 | 31.82 |

>> Consider $y = a e^{bx}$

$\therefore \quad \log_e y = \log_e a + bx \log_e e$. But $\log_e e = 1$

Denoting $Y = \log_e y$, $A = \log_e a$ we have, $Y = A + bx$ which is a straight line.

The associated normal equations are as follows.

$$\sum Y = nA + b\sum x$$

$$\sum xY = A\sum x + b\sum x^2 \quad (n = 3)$$

The relevant table is as follows.

| $x$ | $y$ | $Y = \log_e y$ | $xY$ | $x^2$ |
|---|---|---|---|---|
| 0 | 8.12 | 2.0943 | 0 | 0 |
| 2 | 10 | 2.3026 | 4.6052 | 4 |
| 4 | 31.82 | 3.4601 | 13.8404 | 16 |
| $\sum x = 6$ | | $\sum Y = 7.8570$ | $\sum xY = 18.4456$ | $\sum x^2 = 20$ |

The normal equations become

$$3A + 6b = 7.8570$$

$$6A + 20b = 18.4456$$

On solving ( *Using calculator* ) we have,

$$A = 1.9361, b = 0.34145 \approx 0.3415$$

$$A = \log_e a = 1.9361 \Rightarrow a = e^A = e^{1.9361} = 6.9317$$

The curve of fit is $y = a\, e^{bx}$

Thus $y = (\mathbf{6.9317})\, e^{\mathbf{0.3415}\, x}$ is the curve of fit.

---

**12.** *Fit an exponential curve of the form $y = a\, e^{bx}$ by the method of least squares for the following data*

| *No. of petals* | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|
| *No. of flowers* | 133 | 55 | 23 | 7 | 2 | 2 |

>> **Note** : *The preliminary steps are to be retraced as in the previous problem. We shall prepare the relevant table with reference to the same.* ( $n = 6$ )

| $x$ | $y$ | $y = \log_e y$ | $xY$ | $x^2$ |
|---|---|---|---|---|
| 5 | 133 | 4.8903 | 24.4515 | 25 |
| 6 | 55 | 4.0073 | 24.0438 | 36 |
| 7 | 23 | 3.1355 | 21.9485 | 49 |
| 8 | 7 | 1.9459 | 15.5672 | 64 |
| 9 | 2 | 0.6931 | 6.2379 | 81 |
| 10 | 2 | 0.6931 | 6.9310 | 100 |
| $\sum x = 45$ | | $\sum Y = 15.3652$ | $\sum xY = 99.1799$ | $\sum x^2 = 355$ |

The normal equations become

$$6A + 45b = 15.3652$$

$$45A + 355b = 99.1799$$

On solving ( *Using calculator* ) we have,

$$A = 9.4433 \quad \text{and} \quad b = -0.9177$$

$$\log_e a = A \quad \Rightarrow \quad a = e^A = e^{9.4433} = 12623.3$$

Thus the required curve of fit is $y = (12623.3)\, e^{-0.9177x}$

---

13. *Fit a least square geometric curve $y = a x^b$ for the following data.*

| $x$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $y$ | 0.5 | 2 | 4.5 | 8 | 12.5 |

>> Consider $y = a x^b$

$\therefore \quad \log_e y = \log_e a + b \log_e x$ and let $Y = \log_e y$, $A = \log_e a$, $X = \log_e x$.

The normal equations associated with $Y = A + bX$ are as follows.

$$\sum Y = nA + b \sum X$$

$$\sum XY = A \sum X + b \sum x^2 \qquad (n = 5)$$

The relevant table is as follows.

| $x$ | $y$ | $X = \log_e x$ | $Y = \log_e y$ | $XY$ | $X^2$ |
|---|---|---|---|---|---|
| 1 | 0.5 | 0 | −0.6931 | 0 | 0 |
| 2 | 2 | 0.6931 | 0.6931 | 0.4804 | 0.4804 |
| 3 | 4.5 | 1.0986 | 1.5041 | 1.6524 | 1.2069 |
| 4 | 8 | 1.3863 | 2.0794 | 2.8827 | 1.9218 |
| 5 | 12.5 | 1.6094 | 2.5257 | 4.0649 | 2.5902 |
| | | $\sum X = 4.7874$ | $\sum Y = 6.1092$ | $\sum XY = 9.0804$ | $\sum X^2 = 6.1993$ |

The normal equations become

$$5A + 4.7874b = 6.1092$$

$$4.7874A + 6.1993b = 9.0804$$

On solving (*Using calculator*) we have, $A = -0.69315$, $b = 2$

$$\log_e a = A \quad \Rightarrow a = e^A = e^{-0.69315} = 0.5$$

Thus $\quad y = 0.5\, x^2$ is the required curve of fit.

---

**14.** *An experiment on lifetime t of cutting tool at different cutting speeds v (units) are given below.*

| *Speed ( v )* | 350 | 400 | 500 | 600 |
|---|---|---|---|---|
| *Life ( t )* | 61 | 26 | 7 | 2.6 |

*Fit a relation of the form* $v = a\,t^b$

>> Consider $v = a\,t^b$

$\therefore \quad \log_e v = \log_e a + b \log_e t$

Let $V = \log_e v$, $A = \log_e a$, $T = \log_e t$ and hence we have the equation $V = A + b\,T$. The associated normal equations are as follows.

$$\sum V = nA + b\sum T$$

$$\sum VT = A\sum T + b\sum T^2 \qquad (n = 4)$$

The relevant table is as follows.

| $v$ | $t$ | $V = \log_e v$ | $T = \log_e t$ | $VT$ | $T^2$ |
|---|---|---|---|---|---|
| 350 | 61 | 5.8579 | 4.1109 | 24.0812 | 16.8995 |
| 400 | 26 | 5.9915 | 3.2581 | 19.5209 | 10.6152 |
| 500 | 7 | 6.2146 | 1.9459 | 12.093 | 3.7865 |
| 600 | 2.6 | 6.3969 | 0.9555 | 6.1122 | 0.913 |
| | | $\sum V = 24.4609$ | $\sum T = 10.2704$ | $\sum VT = 61.8073$ | $\sum T^2 = 32.2142$ |

The normal equations become

$$4A + 10.2704\,b = 24.4609$$

$$10.2704\,A + 32.2142b = 61.8073$$

On solving ( *Using calculator* ) we have $A = 6.5539$ and $b = -0.1709$

$$A = \log_e a \;\Rightarrow\; a = e^A = e^{6.5539} = 701.9766$$

Thus the required curve of fit is $v = (\mathbf{701.9766})\,t^{-\mathbf{0.1709}}$

**15.** *Fit a curve of the form $y = a\,b^x$ for the data and hence find the estimation for $y$ when $x = 8$.*

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $y$ | 87 | 97 | 113 | 129 | 202 | 195 | 193 |

>> Consider $y = a\,b^x$

$\therefore \quad \log_e y = \log_e a + x \log_e b$

Let $Y = \log_e y$, $A = \log_e a$, $B = \log_e b$ and hence we have the equation $Y = A + B\,x$. The associated normal equations are as follows.

$$\sum Y = nA + B \sum x$$

$$\sum xY = A \sum x + B \sum x^2 \quad (n = 7)$$

The relevant table is as follows.

| $x$ | $y$ | $Y = \log_e y$ | $xY$ | $x^2$ |
|---|---|---|---|---|
| 1 | 87 | 4.4659 | 4.4659 | 1 |
| 2 | 97 | 4.5747 | 9.1494 | 4 |
| 3 | 113 | 4.7274 | 14.1822 | 9 |
| 4 | 129 | 4.8598 | 19.4392 | 16 |
| 5 | 202 | 5.3083 | 26.5415 | 25 |
| 6 | 195 | 5.2730 | 31.6380 | 36 |
| 7 | 193 | 5.2627 | 36.8389 | 49 |
| $\sum x = 28$ | | $\sum Y = 34.4718$ | $\sum xY = 142.2551$ | $\sum x^2 = 140$ |

The normal equations become

$$7A + 28B = 34.4718$$

$$28A + 140B = 142.2551$$

On solving ( *Using calculator* ) we have $A = 4.3, B = 0.156$

$$A = \log_e a \quad \Rightarrow \quad a = e^A = e^{4.3} = 73.7$$

$$B = \log_e b \quad \Rightarrow \quad b = e^B = e^{0.156} = 1.169$$

Thus the required curve of fit is $\mathbf{y = 73.7\,(1.169)^x}$

Also **when** $x = 8$, $y = 73.7\,(1.169)^8 = \mathbf{257.03}$

---

## EXERCISES

*Find the equation of the best fitting straight line for the following data* [ 1 to 6 ]

**1.**

| | | | | | |
|---|---|---|---|---|---|
| $x$ | 1 | 2 | 3 | 4 | 5 |
| $y$ | 14 | 13 | 9 | 5 | 2 |

**2.**

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
| $y$ | 9 | 8 | 24 | 28 | 26 | 20 |

**3.**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| $y$ | 2 | 1 | 3 | 2 | 4 | 3 | 5 |

**4.**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ | 62 | 64 | 65 | 69 | 70 | 71 | 72 |
| $y$ | 65.7 | 66.8 | 67.2 | 69.3 | 69.8 | 70.5 | 70.9 |

**5.**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| $y$ | 80 | 90 | 92 | 83 | 94 | 99 | 92 |

**6.**

| | | | | | |
|---|---|---|---|---|---|
| Year ($x$) | 1911 | 1921 | 1931 | 1941 | 1951 |
| Production ($y$) (in thousand tons) | 8 | 10 | 12 | 10 | 6 |

*Fit a parabola of second degree for the following data* [ 7 to 10 ]

**7.**

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| $y$ | 14 | 18 | 23 | 29 | 36 | 40 | 46 |

**8.**

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ | 10 | 20 | 30 | 40 | 50 | 60 |
| $y$ | 157 | 179 | 210 | 252 | 302 | 361 |

**9.**

| | | | | | |
|---|---|---|---|---|---|
| $x$ | 0 | 1 | 2 | 3 | 4 |
| $y$ | 1 | 5 | 10 | 22 | 38 |

**10.**

| | | | | | |
|---|---|---|---|---|---|
| $x$ | 1 | 2 | 3 | 4 | 5 |
| $y$ | 25 | 28 | 33 | 39 | 46 |

**11.** Fit a curve of the form $y = a\,b^x$ for the data and hence estimate $y$ when $x = 8$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| $y$ | 32 | 47 | 65 | 92 | 132 | 190 | 275 |

12. Fit a curve of the form $y = a\,x^b$ for the data

| $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $y$ | 2.98 | 4.26 | 5.21 | 6.1 | 6.8 | 7.5 |

13. Find the equation of the best fitting curve in the form $y = a\,e^{bx}$ for the data

| $x$ | 0 | 2 | 4 |
|---|---|---|---|
| $y$ | 5.02 | 10 | 31.62 |

14. Find a law of the form $y = a + bx^2$ for the data by first deriving the normal equations.

| $x$ | 10 | 20 | 30 | 40 | 50 |
|---|---|---|---|---|---|
| $y$ | 8 | 10 | 15 | 21 | 30 |

15. Find a law of the form $V = a + (b/A)$ for the following data and hence find $V$ when $A = 12$ (*Hint*: Take $1/A = x$ )

| $V$ | 50 | 47 | 46 | 45 | 44 |
|---|---|---|---|---|---|
| $A$ | 2 | 3 | 4 | 6 | 10 |

## ANSWERS

1. $y = -3.2\,x + 18.2$
2. $y = 3.23\,x + 11.096$
3. $y = 0.5\,x + 1.36$
4. $y = 0.52\,x + 33.46$
5. $y = 2x + 82$
6. $y = 0.16\,x - 302.56$
7. $y = 0.083\,x^2 + 4.96\,x + 13.46$
8. $y = 0.046\,x^2 - 0.84\,x + 143.67$
9. $y = 2.23\,x^2 + 0.18\,x + 1.46$
10. $y = 0.64\,x^2 + 1.46\,x + 2.78$
11. $y = 32.15\,(1.43)^x$, 562
12. $y = 2.98\,x^{0.31}$
13. $y = 4.64\,e^{0.46\,x}$
14. $y = 6.3 + 0.009\,x^2$
15. $V = 42.4 + (14.82/A)$ ; 43.635

## 4.3 Optimization

***Optimization*** is a technique of obtaining the best results under the prevailing/given circumstances. Optimization means maximization or minimization.

***Programming*** is a mathematical technique to determine the optimum use of the limited available resources.

***Linear programming*** is a decision making technique under the given constraints on the condition that the relationship among the variables involved is linear. A general relationship among the variables involved is called *objective function*. The variables involved are called *decision variables*.

***Solution by linear programming methods aims at optimization.***

We first present the mathematical formulation of a Linear Programming Problem (LPP) and proceed to discuss the solution of LPP by *Graphical method* and *Simplex method*

### 4.31 Mathematical formulation of a linear programming problem

A linear programming problem is mathematically formulated by first identifying a set of variables $x_1, x_2, \ldots x_n$ which are subject to certain linear conditions known as *constraints* written in the form of inequalities.

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq (\geq) b_1$$

$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \leq (\geq) b_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq (\geq) b_n$$

where the coefficients $a_{ij}$, $b_i$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$ are constants and $x_1 \geq 0,\ x_2 \geq 0 \cdots x_n \geq 0$.

The set of inequalities can be put in the matrix form $AX \leq B$ or $AX \geq B$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$$

The objective function involving the variables $x_1, x_2, \ldots x_n$ along with the given constants $c_1, c_2 \ldots c_n$ will be a linear function of the form

$$Z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

**The optimization (maximization or minimization) of the objective function Z subject to the constraints $AX \leq B$ or $AX \geq B$ is the mathematical formulation of a LPP.**

A set of real values $X = (x_1, x_2, \ldots x_n)$ which satisfies the constraint $AX \leq (\geq) B$ is called *solution*.

A set of real values $x_i$ which satisfies the constraints and also satisfy non negativity constraints $x_i \geq 0$ is called ***feasible solution.***

A set of real values $x_i$ which satisfies the constraints along with non negativity restrictions and optimizes the objective function is called ***optimal solution.*** In otherwords feasible solution optimizing the objective function is called optimal solution.

**Note** :

1. *An LPP can have many optimal solutions.*
2. *If the optimal value of the objective function is infinity then the LPP is said to have unbounded solution. Also an LPP may not possess any feasible solution.*

## 4.32 Graphical method of solving an LPP

LPP involved with only two decision variables can be solved in this method.
The method is illustrated step wise when the problem is mathematically formulated.

- The constraints are considered in the form of equalities. Obviously these represent straight lines since there are only two decision variables.
- The equations are put in the form $\frac{x}{a} + \frac{y}{b} = 1$ which graphically represents straight line passing through the points $(a, 0)$ and $(0, b)$.
- These lines along with the co-ordinate axes forms the boundary of the region known as the *feasible region* and the figure so formed by the vertices is called the *convex polygon*.
- The value of the objective function $Z$ is found at all these vertices.
- The optimum/extreme values of $Z$ ( *maximum or minimum* ) among these values corresponding to the values of the decision variables is the required optimal solution of the LPP.

## WORKED PROBLEMS

16. *Maximize* $z = x + 1.5y$ *given* $x \geq 0,\ y \geq 0$ *subject to the constraints :* $x + 2y \leq 160,\ 3x + 2y \leq 240$ *by graphical method.*

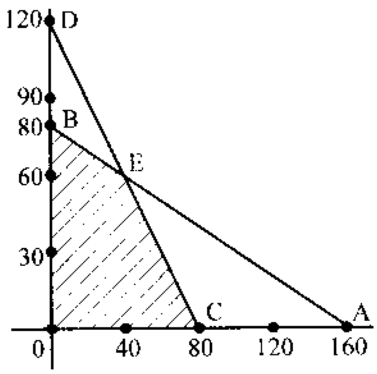
>> Let us consider the equations,

$$x + 2y = 160 \quad : \quad 3x + 2y = 240$$

$$\Rightarrow \quad \frac{x}{160} + \frac{y}{80} = 1 \quad \ldots(1) \quad : \quad \frac{x}{80} + \frac{y}{120} = 1 \quad \ldots(2)$$

Let (1) represent the straight line $AB$ joining the points $A(160, 0)$, $B(0, 80)$ and (2) represent the straight line $CD$ joining $C(80, 0)$, $D(0, 120)$.

We draw the lines $AB$ and $CD$ in the $XOY$ plane.




Shaded portion is the feasible region and $O\,C\,E\,B$ is the convex polygon. The point $E$ is the point of intersection of the lines $AB$ and $CD$. We can obtain this point by solving simultaneously the system of equations :

$x + 2y = 160$ and $3x + 2y = 240$ On solving we obtain $E(x, y) = (40, 60)$

The value of the objective function at the corners of the convex polygon $O\,C\,E\,B$ are tabulated.

| Corner | Value of $Z = x + 1.5y$ |
|---|---|
| $O(0, 0)$ | 0 |
| $C(80, 0)$ | 80 |
| $E(40, 60)$ | 130 |
| $B(0, 80)$ | 120 |

**Thus Z is maximum at $E(x, y) = (40, 60)$ & has the maximum value equal to 130.**

17. *Minimize* $z = 5x + 4y$ *subject to the constraints:* $x + 2y \geq 10$, $x + y \geq 8$, $2x + y \geq 12$, $x \geq 0$, $y \geq 0$ *by graphical method.*

>> Let us consider the equations :

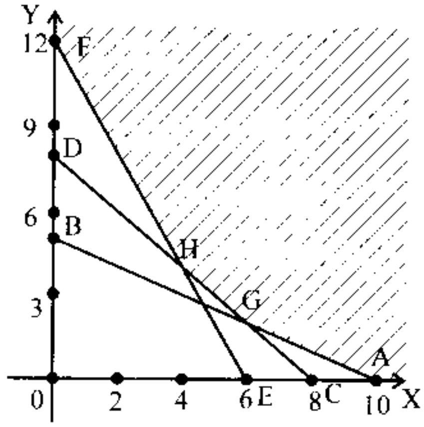
$$x + 2y = 10 \quad ; \quad x + y = 8 \quad ; \quad 2x + y = 12$$

$$\Rightarrow \quad \frac{x}{10} + \frac{y}{5} = 1 \quad \ldots(1) \quad ; \quad \frac{x}{8} + \frac{y}{8} = 1 \quad \ldots(2) \quad ; \quad \frac{x}{6} + \frac{y}{12} = 1 \quad \ldots(3)$$

Let (1), (2), (3) respectively represent the straight lines $AB$, $CD$, $EF$ where we have,

$A = (10, 0)$, $B = (0, 5)$ ; $C = (8, 0)$, $D = (0, 8)$ ; $E = (6, 0)$, $F = (0, 12)$

We draw these lines in the $XOY$ plane.




Shaded portion is the feasible region. $G$ is the point of intersection of the lines $AB$ & $CD$ and $H$ is the point of intersection of the lines $CD$ & $EF$.

By solving $x + 2y = 10$ and $x + y = 8$ we get $G = (6, 2)$.

Also by solving $x + y = 8$, $2x + y = 12$ we get $H = (4, 4)$.

Here the feasible region is unbounded and we tabulate the value of the objective function corresponding to the points $A$, $G$, $H$, $F$.

| Corner | Value of $Z = 5x + 4y$ |
|---|---|
| $A(10, 0)$ | 50 |
| $G(6, 2)$ | 38 |
| $H(4, 4)$ | **36** |
| $F(0, 12)$ | 48 |

**Thus $Z$ is minimum for $x = 4$, $y = 4$ and the minimum value is 36.**

---

**18.** *Use the graphical method to maximize* $z = 3x + 4y$ *subject to the constraints,* $2x + y \le 40$, $2x + 5y \le 180$, $x \ge 0$, $y \ge 0$.

>> Let us consider the equations :

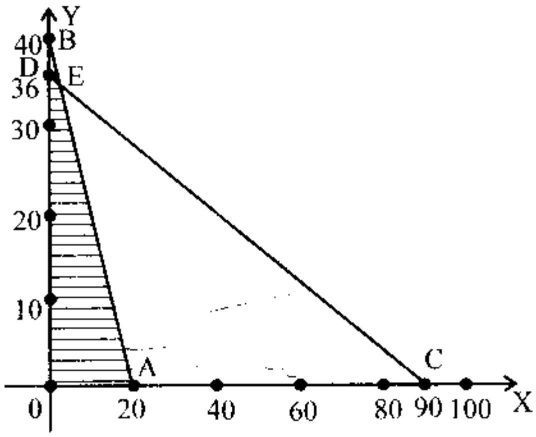
$2x + y = 40$ ; $2x + 5y = 180$

$\Rightarrow \quad \dfrac{x}{20} + \dfrac{y}{40} = 1 \quad \ldots (1) \quad ; \quad \dfrac{x}{90} + \dfrac{y}{36} = 1 \quad \ldots (2)$

Let (1) and (2) represent the straight lines $AB$ and $CD$ respectively where we have

$$A = (20, 0), \; B = (0, 40) \; ; \; C = (90, 0), \; D = (0, 36)$$

We draw these lines in $XOY$ plane.




Shaded portion is the feasible region and $OAED$ is the convex polygon. The point $E$ being the point of intersection of lines $AB$ and $CD$ is obtained by solving the equations : $2x + y = 40, \; 2x + 5y = 180. \; E(x, y) = (2.5, 35)$

The value of the objective function at the corners of the convex polygon $OAED$ are tabulated.

| Corner | Value of $Z = 3x + 4y$ |
|---|---|
| $O(0, 0)$ | 0 |
| $A(20, 0)$ | 60 |
| $E(2.5, 35)$ | **147.5** |
| $D(0, 36)$ | 144 |

**Thus $Z$ is maximum at the vertex (2.5, 35) & has the maximum value equal to 147.5**

---

**19.** *Use the graphical method to minimize* $z = 20x_1 + 10x_2$ *subject to the constraints,* $x_1 + 2x_2 \le 40, \; 3x_1 + x_2 \ge 30, \; 4x_1 + 3x_2 \ge 60, \; x_1 \ge 0, \; x_2 \ge 0.$

>> Let us consider the equations

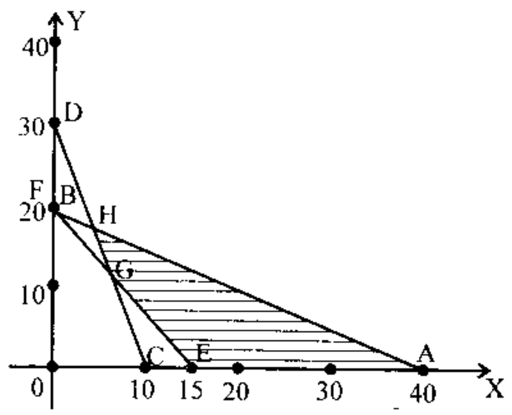
$$x_1 + 2x_2 = 40 \quad ; \quad 3x_1 + x_2 = 30 \quad ; \quad 4x_1 + 3x_2 = 60$$

$$\Rightarrow \quad \frac{x_1}{40} + \frac{x_2}{20} = 1 \quad \ldots (1) \; ; \quad \frac{x_1}{10} + \frac{x_2}{30} = 1 \quad \ldots (2) \; ; \quad \frac{x_1}{15} + \frac{x_2}{20} = 1 \quad \ldots (3)$$

Let (1), (2), (3) represent the straight lines $AB$, $CD$, $EF$ respectively where we have

$A = (40, 0)$, $B = (0, 20)$; $C = (10, 0)$, $D = (0, 30)$; $E = (15, 0)$, $F = (0, 20)$

We draw these lines in the $XOY$ plane.




Shaded portion is the feasible region and $E\ A\ H\ G$ is the associated convex polygon. $G$ is the point of intersection of the lines $CD$ and $EF$. $H$ is the point of intersection of the lines $AB$ and $CD$.

On solving $3x_1 + x_2 = 30$ & $4x_1 + 3x_2 = 60$ we get $G(x_1, x_2) = (6, 12)$

On solving $3x_1 + x_2 = 30$ & $x_1 + 2x_2 = 40$ we get $H(x_1, x_2) = (4, 18)$

The value of the objective function at the corners of the convex polygon $EAHG$ are tabulated.

| Corner | Value of $Z = 20x_1 + 10x_2$ |
|---|---|
| $E(15, 0)$ | 300 |
| $A(40, 0)$ | 800 |
| $H(4, 18)$ | 260 |
| $G(6, 12)$ | **240** |

**Thus $Z$ is minimum at the vertex $(6, 12)$ and the minimum value is 240.**

---

**20.** *Minimize $Z = 30x + 20y$ subject to the constraints, $x - y \le 1$, $x + y \ge 3$, $y \le 4$ and $x \ge 0$, $y \ge 0$ by graphical method.*

>> Let us consider the equations

$x - y = 1$ ; $x + y = 3$ ; $y = 4$

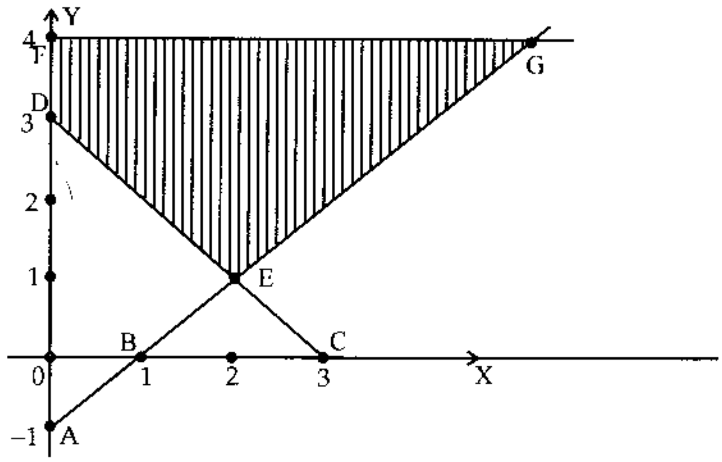
$\Rightarrow \quad \dfrac{x}{1} + \dfrac{y}{-1} = 1 \quad \ldots (1)$ ; $\dfrac{x}{3} + \dfrac{y}{3} = 1 \quad \ldots (2)$ ; $y = 4$

Let (1) and (2) represent the straight lines $AB$ and $CD$ where we have,

$A = (1, 0)$, $B = (0, -1)$ ; $C = (3, 0)$, $D = (0, 3)$

$y = 4$ is a straight line parallel to the X-axis.

We draw these lines.




Shaded portion is the feasible region and $EDFG$ is the associated convex polygon. $E$ is the point of intersection of $AB$ and $CD$. On solving $x - y = 1$ and $x + y = 3$ we get $E(x, y) = (2, 1)$. Also $F(x, y) = (0, 4)$

G is obtained by solving $y = 4$ and $x - y = 1$. That is $G(x, y) = (5, 4)$

The value of the objective function $Z$ at the corners of the convex polygon are tabulated.

| Corner | Value of $Z = 30x + 20y$ |
|---|---|
| $E(2, 1)$ | 80 |
| $D(0, 3)$ | **60** |
| $F(0, 4)$ | 80 |
| $G(5, 4)$ | 230 |

**Thus $Z$ is minimum at the vertex $(0, 3)$ and the minimum value is 60.**

---

**21.** *Use graphical method to solve the following LPP. Minimize $Z = 8x + 5y$ subject to $6x + y \geq 21$, $x + 3y \geq 12$, $x + y \leq 10$ & $x \geq 0$, $y \geq 0$.*

>> Let us consider the equations

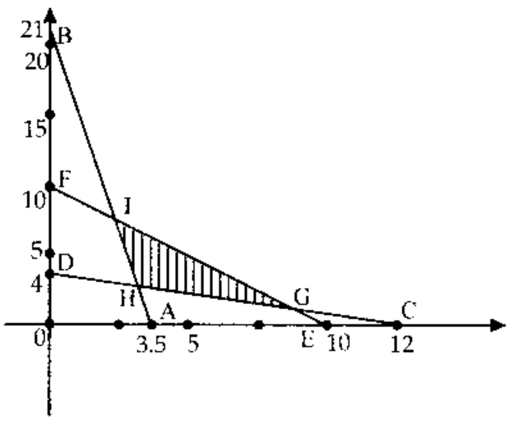
$6x + y = 21$ ; $x + 3y = 12$ ; $x + y = 10$

$$\Rightarrow \quad \frac{x}{3.5} + \frac{y}{21} = 1 \quad \ldots(1) \; ; \quad \frac{x}{12} + \frac{y}{4} = 1 \quad \ldots(2) \; ; \quad \frac{x}{10} + \frac{y}{10} = 1 \quad \ldots(3)$$

Let (1) (2) and (3) respectively represent the straight lines $AB$, $CD$ and $EF$ where we have,

$A = (3.5, 0)$, $B = (0, 21)$ ; $C = (12, 0)$, $D = (0, 4)$ ; $E = (10, 0)$, $F = (0, 10)$

We draw these lines in the $XOY$ plane.




Shaded portion is the feasible region and $G, H, I$ are the vertices of the convex polygon which is a triangle.

On solving : $x + 3y = 12$ and $x + y = 10$ we get $G(x, y) = (9, 1)$

$6x + y = 21$ and $x + 3y = 12$ we get $H(x, y) = (3, 3)$

$x + y = 10$ and $6x + y = 21$ we get $I(x, y) = (2.2, 7.8)$

The value of the objective function at these points are tabulated.

| Corner | Value of $Z = 8x + 5y$ |
|---|---|
| $G(9, 1)$ | 77 |
| $H(3, 3)$ | **39** |
| $I(2.2, 7.8)$ | 56.6 |

**Thus the minimum value of Z is 39 at the vertex (3, 3).**

---

22. *Show that the following LPP does not have any feasible solution.*

*Objective function for maximization :* $Z = 20x + 30y$.

*Constraints :* $3x + 4y \leq 24$, $7x + 9y \geq 63$, $x \geq 0$, $y \geq 0$ *( use graphical method )*

>> Let us consider the equations

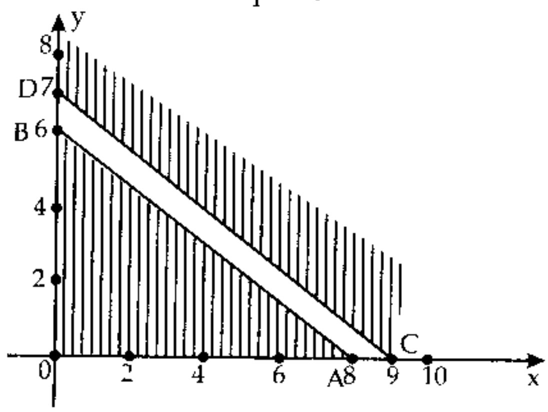
$3x + 4y = 24$ ; $7x + 9y = 63$

$\Rightarrow \quad \frac{x}{8} + \frac{y}{6} = 1 \quad \ldots (1) \quad ; \quad \frac{x}{9} + \frac{y}{7} = 1 \quad \ldots (2)$

Let (1) and (2) respectively represent the straight lines $AB$ and $CD$ where we have

$$A = (8, 0), \ B = (0, 6) \ ; \ C = (9, 0), \ D = (0, 7)$$

The straight lines are drawn in the $XOY$ plane.




It is evident that there is no feasible region.

**Thus we conclude that the LPP does not have any feasible solution.**

---

23. *Show that the following LPP has unbounded optimal solution. Objective function to be maximized :* $Z = 8x + 5y$ *Constraints :* $x - 2y \leq 12$, $x + 2y \geq 20$, $x \geq 0$, $y \geq 0$.

>> Let us consider the equations.

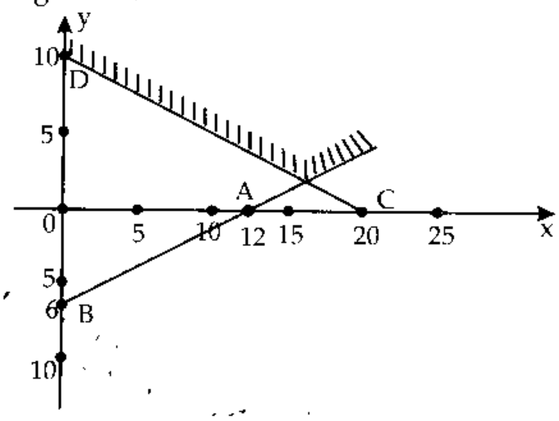
$$x - 2y = 12 \quad ; \quad x + 2y = 20$$

$$\Rightarrow \quad \frac{x}{12} + \frac{y}{-6} = 1 \quad \ldots (1) \quad ; \quad \frac{x}{20} + \frac{y}{10} = 1 \quad \ldots (2)$$

Let (1) and (2) respectively represent the straight lines $AB$ and $CD$ where we have,

$$A = (12, 0), \ B = (0, -6) \ ; \ C = (20, 0), \ D = (0, 10)$$

Let us draw these straight lines.




Shaded portion is the feasible region and it is evident that this region is unbounded.

**Thus we conclude that the LPP has unbounded optimal solution.**

---

**24.** *Solve the following LPP graphically.*

*Maximize* $Z = 3x_1 + 5x_2$ *subject to* $x_1 + 2x_2 \leq 2000$, $x_1 + x_2 \leq 1500$, $x_2 \leq 600$, $x_1, \; x_2 \geq 0$

>> Let us consider the equations

$$x_1 + 2x_2 = 2000 \quad ; \quad x_1 + x_2 = 1500 \quad ; \quad x_2 = 600$$

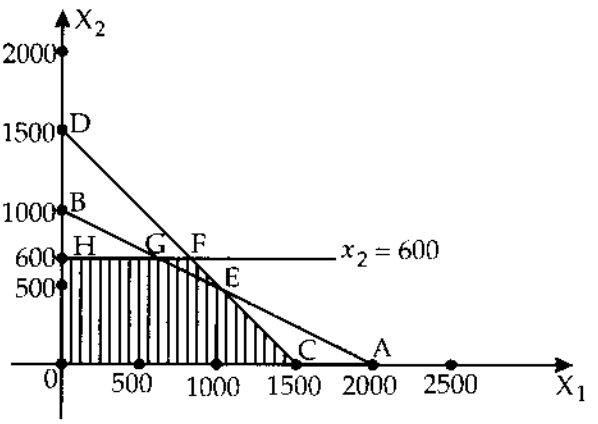
$$\Rightarrow \quad \frac{x_1}{2000} + \frac{x_2}{1000} = 1 \quad \ldots (1) \; ; \quad \frac{x_1}{1500} + \frac{x_2}{1500} = 1 \quad \ldots (2) \; ; \quad x_2 = 600$$

Let (1) and (2) respectively represent the lines $AB$ and $CD$ where we have,

$A = (2000, 0)$, $B = (0, 1000)$ ; $C = (1500, 0)$, $D = (0, 1500)$

$x_2 = 600$ is a line parallel to the $x_1$ axis.

Let us draw these straight lines.




On solving

$x_1 + 2x_2 = 2000$, $x_1 + x_2 = 1500$ we get $E = (1000, 500)$

$x_1 + x_2 = 1500$, $x_2 = 600$ we get $F = (900, 600)$

$x_1 + 2x_2 = 2000$, $x_2 = 600$ we get $G = (800, 600)$

Also we have $C = (1500, 0)$, $H = (0, 600)$

The value of the objective function at these vertices are tabulated. $OCEFGH$ is the convex polygon.

| Corner | Value of $Z = 3x_1 + 5x_2$ |
|---|---|
| $O(0, 0)$ | 0 |
| $C(1500, 0)$ | 4500 |
| $E(1000, 500)$ | 5500 |
| $F(900, 600)$ | **5700** |
| $G(800, 600)$ | 5400 |

**Thus $(Z)_{Max} = 5700$ when $x_1 = 900$, $x_2 = 600$**

25. *Maximize* $z = 50x_1 + 60x_2$ *subject to the constraints :* $2x_1 + 3x_2 \le 1500$, $3x_1 + 2x_2 \le 1500$, $0 \le x_1 \le 400$, $0 \le x_2 \le 400$, $x_1 \ge 0$, $x_2 \ge 0$
( *use graphical method* )

>> Let us consider the equations

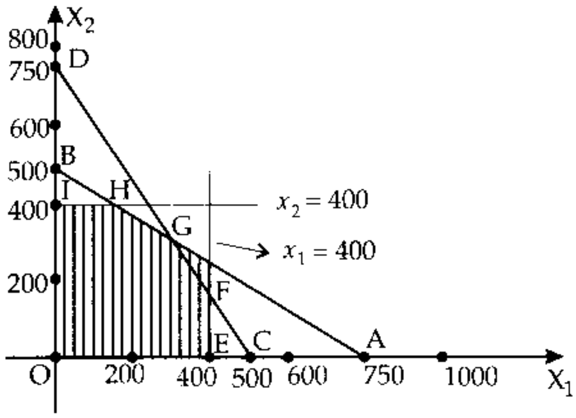
$$2x_1 + 3x_2 = 1500 \quad ; \quad 3x_1 + 2x_2 = 1500,\ x_1 = 400,\ x_2 = 400$$

$$\Rightarrow \quad \frac{x_1}{750} + \frac{x_2}{500} = 1 \quad \ldots(1) \quad ; \quad \frac{x_1}{500} + \frac{x_2}{750} = 1 \quad \ldots(2)$$

Let (1) and (2) respectively represent the straight lines $AB$ and $CD$ where we have,

$$A = (750, 0),\ B = (0, 500) \ ; \ C = (500, 0),\ D = (0, 750)$$

Let us draw these straight lines.




On solving $3x_1 + 2x_2 = 1500$ and $x_1 = 400$ we get $F = (400, 150)$

$2x_1 + 3x_2 = 1500$ and $3x_1 + 2x_2 = 1500$ we get $G = (300, 300)$

$2x_1 + 3x_2 = 1500$ and $x_2 = 400$ we get $H = (150, 400)$

Also $E = (400, 0)$ ; $I = (0, 400)$

The value of the objective function at these vertices are tabulated. *OEFGHI* is the convex polygon.

| Corner | Value of $Z = 50x_1 + 60x_2$ |
|---|---|
| $O(0, 0)$ | 0 |
| $E(400, 0)$ | 20,000 |
| $F(400, 150)$ | 29,000 |
| $G(300, 300)$ | **33,000** |
| $H(150, 400)$ | 31,500 |
| $I(0, 400)$ | 24,000 |

$(Z)_{Max}$ **= 33,000 at** $x_1 = 300$ **and** $x_2 = 300$

---

**26.** *Two spare parts X and Y are to be produced in a batch. Each one has to go through two processes A and B. The time required in hours per unit and the total time available are given.*

| | X | Y | *Total hours available* |
|---|---|---|---|
| *Process A* | 3 | 4 | 48 |
| *Process B* | 9 | 4 | 72 |

*Profits per unit of X and Y are rupees 5 and 6 respectively. Find how many number of spare parts of X and Y are to be produced in this batch to maximize the profit. (Use graphical method)*

>> The mathematical formulation of the LPP is to maximize $Z = 5X + 6Y$ subject to the constraints $3X + 4Y \leq 48$, $9X + 4Y \leq 72$.

Let us consider the equations

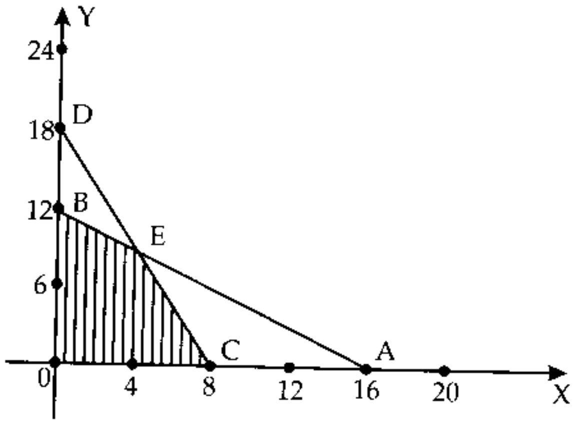
$$3X + 4Y = 48 \quad ; \quad 9X + 4Y = 72$$

$$\Rightarrow \quad \frac{X}{16} + \frac{Y}{12} = 1 \quad ; \quad \frac{X}{8} + \frac{Y}{18} = 1$$

Let *AB* and *CD* represent the straight lines where we have

$$A = (16, 0),\ B = (0, 12)\ ; \quad C = (8, 0),\ D = (0, 18)$$

The straight lines are drawn in the *XOY* plane.




On solving $3X + 4Y = 48$ and $9X + 4Y = 72$ we get $E(X, Y) = (4, 9)$. The value of the objective function at the vertices of the convex polygon $OCEB$ are tabulated.

| Corner | Value of $Z = 5X + 6Y$ |
|---|---|
| $O(0, 0)$ | 0 |
| $C(8, 0)$ | 40 |
| **$E(4, 9)$** | **74** |
| $B(0, 12)$ | 72 |

$(Z)_{Max} = 74$ at $(X, Y) = (4, 9)$

**Thus we can say that 4 Nos. of $X$ and 9 Nos. of $Y$ spare parts are to be produced to get the maximum profit of Rs.74**

---

27. *A chemist wishes to provide for his customers, at the least cost the minimum daily requirements of three vitamins 1, 2 and 3 by using a mixture of two products M and N. The amount of each vitamin in one gram of each product, the cost per gram of each product and the minimum daily requirements are given below.*

| | *Number of units of each vitamin contained in a gram of each product* | | | *Cost per gram of each product* |
|---|---|---|---|---|
| | *Vitamin 1* | *Vitamin 2* | *Vitamin 3* | |
| *Product M* | 6 | 2 | 4 | 20 *Paise* |
| *Product N* | 2 | 2 | 12 | 16 *Paise* |
| *Minimum requirement of each vitamin* | 12 | 8 | 24 | |

*Find the least expensive combination which provides the minimum requirements of these vitamins (use the graphical method).*

>> We shall first mathematically formulate the problem :

Let $x$ = Number of grams of product $M$ in the mixture.

$y$ = Number of grams of product $N$ in the mixture.

From the data the constraints of the L.P.P are

$$6x + 2y \geq 12, \quad 2x + 2y \geq 8, \quad 4x + 12y \geq 24.$$

The problem is to minimize the objective function $Z = 20x + 16y$.

Let us consider the equations,

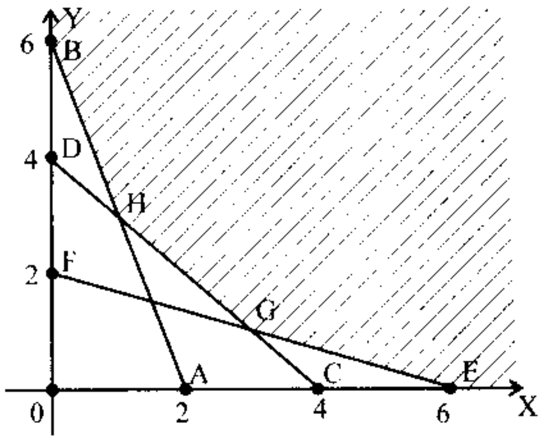
$$6x + 2y = 12, \quad 2x + 2y = 8, \quad 4x + 12y = 24$$

$$\Rightarrow \quad \frac{x}{2} + \frac{y}{6} = 1 \quad \ldots(1) \qquad \frac{x}{4} + \frac{y}{4} = 1 \quad \ldots(2) \qquad \frac{x}{6} + \frac{y}{2} = 1 \quad \ldots(3)$$

Let (1), (2), (3) represent the straight lines $AB$, $CD$, $EF$ respectively where we have

$A = (2, 0)$, $B = (0, 6)$, $C = (4, 0)$ ; $D = (0, 4)$, $E = (6, 0)$, $F = (0, 2)$

We now draw these lines in $XOY$ plane.




Shaded portion is the feasible region which is unbounded. The point $G$ is the point of intersection of the lines $CD$ and $EF$. The point $H$ is the intersection of the lines $CD$ and $AB$.

The point G $(x, y) = (3, 1)$ is obtained by solving $2x + 2y = 8$ and $4x + 12y = 24$..

The point $H(x, y) = (1, 3)$ is obtained by solving $2x + 2y = 8$ and $6x + 2y = 12$..

The value of the objective function at the corners $E$, $G$, $H$, $B$ are tabulated.

| Points | Value of $Z = 20x + 16y$ |
|---|---|
| $E(6, 0)$ | 120 |
| $G(3, 1)$ | 76 |
| $H(1, 3)$ | 68 |
| $B(0, 6)$ | 96 |

Minimum cost is 68 paise. **Thus the least expensive combination is 1 part of the product $M$ and 3 parts of the product $N$ at the minimum cost of 68 paise.**

---

28. *In the production of two type of watches a factory uses three machines A, B, C. The time required for each watch on each machine and the maximum time available on each machine is given below.*

| *Machine* | *Time required* | | *Maximum time available (In hours)* |
|---|---|---|---|
| | *Watch I* | *Watch II* | |
| *A* | 6 | 8 | 380 |
| *B* | 8 | 4 | 300 |
| *C* | 12 | 4 | 404 |

*The profit on watch I is Rs.50 and on watch II is Rs.30. Find what combination should be produced for the maximum profit. What is the maximum profit (use graphical method).*

>> We shall first mathematically formulate the problem.

Let $x$ = number of type $I$ watches to be produced and $y$ = number of type $II$ watches to be produced.

The objective function $Z = 50x + 30y$, subject to the following constraints has to be maximized.

$$6x + 8y \le 380, \quad 8x + 4y \le 300, \quad 12x + 4y \le 404$$

Let us consider the equations

$$6x + 8y = 380 \quad ; \quad 8x + 4y = 300 \quad ; \quad 12x + 4y = 404$$

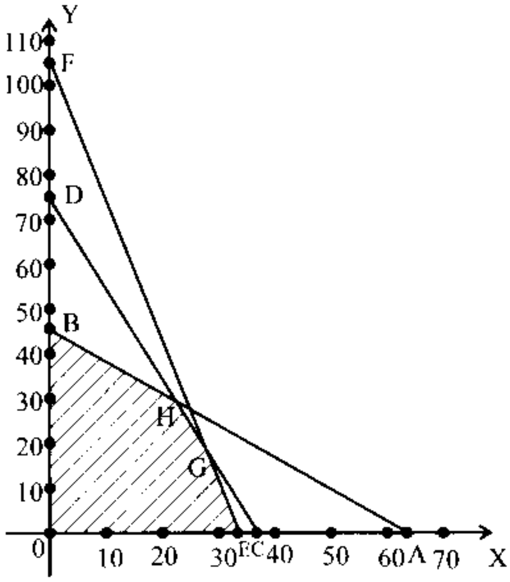
$$\Rightarrow \quad \frac{x}{63.3} + \frac{y}{47.5} = 1 \quad \ldots (1) \; ; \quad \frac{x}{37.5} + \frac{y}{75} = 1 \quad \ldots (2) \; ; \quad \frac{x}{33.7} + \frac{y}{101} = 1 \quad \ldots (3)$$

Let (1), (2), (3) represent the straight lines $AB$, $CD$, $EF$ respectively where we have

$A = (63.3, 0)$ $C = (37.5, 0)$ $E = (33.7, 0)$

$B = (0, 47.5)$ $D = (0, 75)$ $F = (0, 101)$

We draw these lines in the *XOY* plane.




Shaded portion is the feasible region and the associated convex polygon is *O E G H B*. *G* is the point of intersection of the lines *CD* and *EF* & *H* is that of the intersection of the lines *AB* and *CD*.

$G(x, y) = (26, 23)$ is obtained by solving $8x + 4y = 300$ and $12x + 4y = 404$

$H(x, y) = (22, 31)$ is obtained by solving $8x + 4y = 300$ and $6x + 8y = 380$.

The value of the objective function at the corners of the convex polygon *OEGHB* are tabulated.

| Corner | Value of $Z = 50x + 30y$ |
|---|---|
| $O(0, 0)$ | 0 |
| $E(33.7, 0)$ | 1685 |
| $G(26, 23)$ | 1990 |
| $H(22, 31)$ | **2030** |
| $B(0, 47.5)$ | 1425 |

*Z* is maximum at the vertex $H(22, 31)$ and the maximum value of *Z* is 2030.

It can be seen from the figure that *H* is the farthest point from the origin.

**Thus the factory has to produce 22 watches of type *I* and 31 watches of type *II* to obtain the maximum profit of Rs.2030.**

**29.** *A company produces two types of food stuffs $F_1$ and $F_2$ which contains three vitamins $V_1$, $V_2$, $V_3$ respectively. Minimum daily requirement of these vitamins are 1mg, 50mg and 10mg respectively. Suppose that the food stuff $F_1$ contains 1mg of $V_1$, 100mg of $V_2$ and 10mg of $V_3$ where as $F_2$ contains 1mg of $V_1$, 10mg of $V_2$ and 100mg of $V_3$. If the cost of food stuff $F_1$ is Rs.2 and that of $F_2$ is Rs.3, find the minimum cost diet that would supply the body the minimum requirement of each vitamin by applying graphical method.*

>> Let $x$ and $y$ be the units of food stuff $F_1$ and $F_2$.

The objective function $Z$ is the total cost of a diet containing $x$ units of $F_1$ and $y$ units of $F_2$. Hence $Z = 2x + 3y$.

Each food stuff contains $V_1$, $V_2$, $V_3$ and their associated inequalities form the constraints given by

$$x + y \geq 1, \quad 100x + 10y \geq 50, \quad 10x + 100y \geq 10 \quad \text{where } x \geq 0 \;\; y \geq 0.$$

The L.P.P is to minimize $Z$, subject to the constraints formulated.

Let us consider the equations.

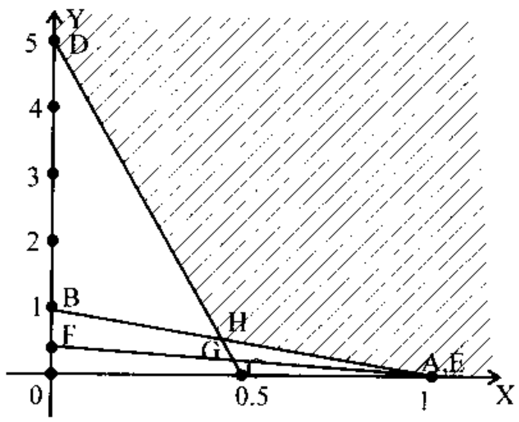
$$x + y = 1 \quad ; \quad 100x + 10y = 50 \quad ; \quad 10x + 100y = 10$$

$$\Rightarrow \quad \frac{x}{1} + \frac{y}{1} = 1 \quad \ldots (1) \quad ; \quad \frac{x}{0.5} + \frac{y}{5} = 1 \quad \ldots (2) \quad ; \quad \frac{x}{1} + \frac{y}{0.1} = 1 \quad \ldots (3)$$

Let (1), (2), (3) represent the straight lines $AB$, $CD$, $EF$ respectively where we have

$A = (1, 0)$, $B = (0, 1)$, $C = (0.5, 0)$, $D = (0, 5)$ ; $E = (1, 0)$, $F = (0, 0.1)$

We draw these lines in the $XOY$ plane.




Shaded portion is the feasible region which is unbounded. $H$ is the point of intersection of the line $AB$ and $CD$. $H(x, y) = (4/9, 5/9)$ obtained by solving $x + y = 1$, $100x + 10y = 50$.

The value of $Z = 2x + 3y$ at $x = 4/9$, $y = 5/9$ is $23/9$. It can be easily seen that the value of $Z$ at $A(1, 0)$ is 2 and the value of $Z$ at $D(0, 5)$ is 15.

$\therefore$ $Z$ is minimum at $A(1, 0)$ and the minimum value is 2.

**Thus one unit of food stuff $F_1$ and no unit of $F_2$ is the least expensive combination and the cost is Rs.2.**

---

**30.** *A factory uses 3 types of machines to produce two types of electronic gadgets. The first gadget requires in hours 12, 4 and 2 respectively on the three types of machines. The second gadget requires in hours 6, 10 and 3 on the machines respectively. The total available time in hours respectively on the machines are 6000, 4000, 1800. If the two types of gadgets respectively fetches a profit of rupees 400 and 1000 find the number of gadgets of each type to be produced for getting the maximum profit.*

>> Let $x$ be the number of gadget of type-1 and $y$ be the number of gadget of type-2.

By data the constraints are as follows.

$$12x + 6y \le 6000\,,\quad 4x + 10y \le 4000\,,\quad 2x + 3y \le 1800$$

The objective function need to be maximized is $Z = 400x + 1000y$

Let us consider the equations.

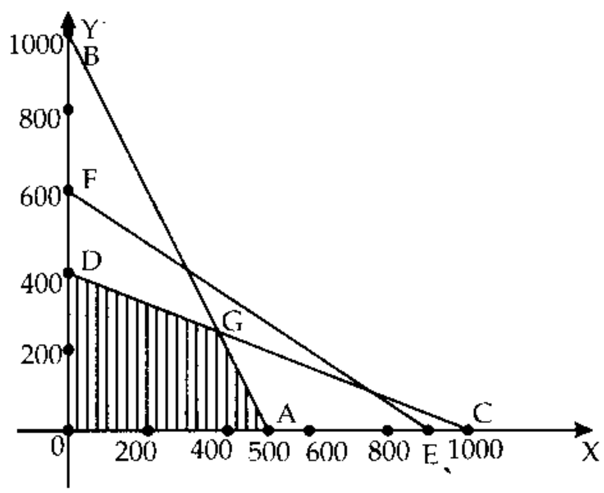
$$12x + 6y = 6000\ ;\quad 4x + 10y = 4000\ ;\quad 2x + 3y = 1800\ ;\quad x \ge 0\,,\quad y \ge 0$$

$$\Rightarrow \quad \frac{x}{500} + \frac{y}{1000} = 1\ ;\quad \frac{x}{1000} + \frac{y}{400} = 1\ ;\quad \frac{x}{900} + \frac{y}{600} = 1$$

Let $AB$, $CD$, $EF$ respectively represent these three lines where we have,

$A = (500, 0)$ $\quad C = (1000, 0)$ $\quad E = (900, 0)$

$B = (0, 1000)$ $\quad D = (0, 400)$ $\quad F = (0, 600)$



The lines are drawn in the $XOY$ plane.

Let $G(x, y)$ be the point of intersection of the lines $AB$ and $CD$.

On solving $12x + 6y = 6000$ and $4x + 10y = 4000$ we get $G(x, y) = (375, 250)$.

The value of the objective function at the corners of the convex polygen $OAGD$ are tabulated.

| Corner | Value of $Z = 400x + 1000y$ |
|---|---|
| $O(0, 0)$ | 0 |
| $A(500, 0)$ | 2,00,000 |
| $G(375, 250)$ | 4,00,000 |
| $D(0, 400)$ | 4,00,000 |

It can be seen that the L.P.P has two optimal solutions.

**Thus we conclude that the factory has to produce 375 gadgets of type-1 and 250 gadgets of type-2 or 400 gadgets of type-2 only to get the maximum profit.**

## 4.33 Preamble for the Simplex Method

Simplex method is an effecient algebraic method to solve a L.P.P by systematic procedure and hence an algorithm can be evolved called the *simplex algorithm*.

In this method it is necessary that all the constraints in the inequality form is converted into equality form thus arriving at a system of algebraic equations and we are familiar with the vairous types of solution of a system of algebraic equations.

[ *Refer Unit-VII in Volume-1 book* ]

If the constraint is involved with $\leq$ we add a non zero variable $s_1$ *(say)* $\geq 0$ to the L.H.S to make it an equality and the same variable is called *slack variable*.

[Ex : $7 + 4 < 12$ ; $7 + 4 + s_1 = 12$, $s_1$ being 1]

If the constraint is involved with $\geq$ we subtract a non zero variable $s_2$ *(say)* $\geq 0$ in the L.H.S to make it an equality and the same variable is called *surplus variable*.

[Ex : $7 + 4 > 8$ ; $7 + 4 - s_2 = 8$, $s_2$ being 3]

L.P.P with all constraints being equalities is called a **standard form of L.P.P**

Given a L.P.P to maximize $Z = ax + by$ subject to the constraints $a_1 x + b_1 y \leq k_1$, $a_2 x + b_2 y \leq k_2$ , $x \geq 0$ , $y \geq 0$, the standard form of L.P.P is to maximize $Z = ax + by$ subject to :

$$a_1 x + b_1 y + s_1 = k_1 , \quad a_2 x + b_2 y + s_2 = k_2 \; ; \; x , y , s_1 , s_2 \geq 0$$

We have a system of two equations in four *unknowns* for obtaining a solution. We know that a system of equations is consistent if it possesses a solution . Suppose we have system of $m$ independent equations in $n$ unknowns then we have the following cases.

(i) If $m = n$, the system has unique solution.

(ii) If $m < n$, the system has many solutions.

(iii) If $m > n$, the system has no solution.

We focus our attention on case - (ii).

In a system of $m$ equations in $n$ unknowns where $n > m$, if $(n - m)$ variables are set to zero then we will have a system of $m$ equations in the remaining $m$ variables. A solution obtained by this way is called *basic solution*. The number of basic solutions will be ${}^nC_m$. The $m$ variables remaining in the system are called *basic variables* or *basis* and $(n - m)$ variables set to zero are called *non basic variables*.

## Basic feasible solution

If the basic solution satisfies non negativity conditions also then the solution is called *basic feasible solution*.

**Simplex method provides basic feasible solution of a L.P.P in a systematic manner.**

**Remark** : *In a solution if all the basic variables are non zero the solution is said to be non degenerate. If one or more basic variables are zero the solution is said to be degenerate.*

## Illustrations

**(1)** Let us consider an L.P.P to maximize

$$P = 2x + 3y + z \quad \text{subject to } 2x + y + 4z = 11 \text{ and } 3x + y + 5z = 14$$

>> Here we have a system of 2 equations in 3 unknowns. That is $m = 2$ and $n = 3$. $n - m = 3 - 2 = 1$. We shall set one variable to zero and solve the system of equations to obtain the basic solutions.

| Case | Variable set to zero | Resulting equations and solution | Basic solution $(x, y, z)$ | Value of $P = 2x + 3y + z$ and remarks |
|---|---|---|---|---|
| (i) | $x = 0$ | $y + 4z = 11$, $y + 5z = 14$; $(y, z) = (-1, 3)$ | $(0, -1, 3)$ | Not a feasible solution |
| (ii) | $y = 0$ | $2x + 4z = 11$, $3x + 5z = 14$; $(x, z) = (1/2, 5/2)$ | $(1/2, 0, 5/2)$ | Feasible solution $P = 7/2$ |
| (iii) | $z = 0$ | $2x + y = 11$, $3x + y = 14$; $(x, y) = (3, 5)$ | $(3, 5, 0)$ | Feasible solution $P = 21$ |

$(x, y, z) = (3, 5, 0)$ **is the optimal basic solution since** $(P)_{Max} = 21$

**(2)** Let us consider a typical L.P.P which we have solved by graphical method. (*Problem*-16) That is maximization of $Z = x + 1.5y$ subject to the constraints $x + 2y \le 160$, $3x + 2y \le 240$, $x, y \ge 0$. We solve this problem like the earlier one by introducing slack variables $s_1, s_2$ for converting the inequality constraints to equality. Equivalently we have the following system of equations.

$$x + 2y + s_1 = 160$$

$$3x + 2y + s_2 = 240 \; ; \; x, y, s_1, s_2 \ge 0$$

This is a system of two equations in four unknowns ( $m = 2, n = 4$ ). We shall set $n - m = 4 - 2 = 2$ variables to zero. There are ${}^nC_m = {}^4C_2 = 6$ ways to choose two variables. The entire process of obtaining the basic solutions and optimal solution is presented in the following table.

| Case | Variables set to zero | Resulting equations and solution | Basic solution $(x, y, s_1, s_2)$ | Value of $Z = x + 1.5y$ and Remark |
|---|---|---|---|---|
| (i) | $x = 0, y = 0$ | $s_1 = 160, s_2 = 240$ | (0, 0, 160, 240) | 0 |
| (ii) | $x = 0, s_1 = 0$ | $2y = 160$, $2y + s_2 = 240$ $(y, s_2) = (80, 80)$ | (0, 80, 0, 80) | 120 |
| (iii) | $x = 0, s_2 = 0$ | $2y + s_1 = 160$, $2y = 240$ $(y, s_1) = (120, -80)$ | (0, 120, −80, 0) | Not a feasible solution $(s_1 < 0)$ |
| (iv) | $y = 0, s_1 = 0$ | $x = 160$, $3x + s_2 = 240$ $(x, s_2) = (160, -240)$ | (160, 0, 0 − 240) | Not a feasible solution $(s_2 < 0)$ |
| (v) | $y = 0, s_2 = 0$ | $x + s_1 = 160$, $3x = 240$ $(x, s_1) = (80, 80)$ | (80, 0, 80, 0) | 80 |
| (vi) | $s_1 = 0, s_2 = 0$ | $x + 2y = 160$, $3x + 2y = 240$ $(x, y) = (40, 60)$ | **(40, 60, 0, 0)** | **130** |

$$\mathbf{(Z)_{Max} = 130 \text{ at } (x, y) = (40, 60)}$$

**Remark** :

1. *We have obtained this result by graphical method.*
2. *This becomes a tedious procedure when the number of variables and constraints are more.*

We proceed to present the simplex method algorithm for solving the LPP which involves sequential steps to achieve the desired result.

## 4.34 Simplex method algorithm

Let us consider the L.P.P with objective function $Z = ax + by$ for maximization subject to the constraints $a_1 x + b_1 y \le k_1$, $a_2 x + b_2 y \le k_2$ ; $x, y \ge 0$

**Step 1** : We convert the linear inequalities into equations by introducing slack variables $s_1, s_2$ and the same are written in a systematic manner as follows.

$$a_1 x + b_1 y + 1.s_1 + 0.s_2 = k_1 \qquad \ldots (1)$$

$$a_2 x + b_2 y + 0.s_1 + 1.s_2 = k_2 \qquad \ldots (2)$$

**Step 2** : These equations are put in a tabular form and is called the initial simplex tableau.

| Non zero variables ( $NZV$ ) | $x$ | $y$ | $s_1$ | $s_2$ | Quantity |
|---|---|---|---|---|---|
| $s_1$ | $a_1$ | $b_1$ | 1 | 0 | $k_1$ |
| $s_2$ | $a_2$ | $b_2$ | 0 | 1 | $k_2$ |
| Indicators ( $\Delta$ ) | $-a$ | $-b$ | 0 | 0 | 0 |

The first two rows in the table contains the various co- efficients in equations (1), (2). The last row contains the coefficients of the variables in the objective function with their sign reversed and are called indicators. This is done only to obtain the positive value of Z. Further when $x = 0$, $y = 0$, $s_1 = 0$, $s_2 = 0$ then $z = 0$.

**Step 3** : (i) We identify the least negative indicator ( *Numerically large* ) and the column in which it is present is called the *pivotal column.*

(ii) Suppose $-a$ is the least indicator, first column is the pivotal column. We examine the ratios $k_1/a_1$ and $k_2/a_2$ and identify the least positive ratio. [*Neglect if there are any negative ratios*]

(iii) Suppose $k_1/a_1$ is least. Then the first row is called the *pivotal row* and $a_1$ is called the *pivot.* Further the variable $s_1$ is replaced by the variable at the top of the pivotal column ( $x$ ). [ $s_1$ *is the departing variable,* $x$ *is the entering variable* ]

**Step 4** : We make the pivot unity by dividing the elements in the pivotal row by the pivot.

The table is now called the *first simplex tableau*.

| NZV | **x** | $y$ | $s_1$ | $s_2$ | Qty |
|---|---|---|---|---|---|
| $x$ | **1** | $b_1/a_1$ | $1/a_1$ | 0 | $k_1/a_1$ |
| $s_2$ | $\mathbf{a_2}$ | $b_2$ | 0 | 1 | $k_2$ |
| $\Delta$ | $\mathbf{-a}$ | $-b$ | 0 | 0 | 0 |

**Step 5** : We reduce the other entries in the pivotal column to zero by elementary transformations.

$$R_2 \to -a_2 R_1 + R_2, \quad R_3 \to a R_1 + R_3$$

These transformations will give rise to the *second simplex tableau*.

| NZV | $x$ | $y$ | $s_1$ | $s_2$ | Qty |
|---|---|---|---|---|---|
| $x$ | 1 | $b_1/a_1$ | $1/a_1$ | 0 | $\mathbf{k_1/a_1}$ |
| $s_2$ | 0 | $b'_2$ | $-a_2/a_1$ | 1 | $\mathbf{k'_2}$ |
| $\Delta$ | 0 | $b'$ | $c'$ | 0 | $\mathbf{d'}$ |

We examine the indicators ( $\Delta$ ) and if there are no negative indicators, the process is completed. We say that Z is maximum at $x = k_1/a_1, y = 0$ and the maximum value of Z is $d'$ provided $b'$ and $c'$ are positive. If we find negative indicators in the second simplex tableau the process will be continued.

## Minimization

A minimizing L.P.P is converted into an equivalent maximization problem. Minimizing the given objective function $P$ is equivalent to maximizing $-P$ under the same constraints and $Min.\ P = -(\ Max\ value\ of\ -P\ )$

## WORKED PROBLEMS

**31.** *Use simplex method to maximize* $Z = x + 1.5y$ *subject to the constraints* $x + 2y \le 160$, $3x + 2y = 240$, $x \ge 0$, $y \ge 0$

>> **Remarks** :

**1.** *We have solved this problem by graphical method and also the solution is obtained while giving preamble for the simplex method as an illustration.*

2. *Now, we solve this problem using the simplex method algorithm in a detailed way as per the steps involved in the method for getting an insight to the method practically.*

***Solution of the subsequent problems are presented in a single table.***

## Solution of the problem

**Step 1 :** Let us introduce slack variables $s_1$ and $s_2$ to the inequalities to write them in the following form .

$$x + 2y + 1.s_1 + 0.s_2 = 160$$
$$3x + 2y + 0.s_1 + 1.s_2 = 240$$

$x + 1.5y + 0.s_1 + 0.s_2 = Z$ is the objective function.

**Step 2 :** We form the initial simplex tableau.

| *NZV* | $x$ | $y$ | $s_1$ | $s_2$ | Qty |
|---|---|---|---|---|---|
| $s_1$ | 1 | **2** | 1 | 0 | 160 |
| $s_2$ | 3 | **2** | 0 | 1 | 240 |
| Indicators ( Δ ) | – 1 | **–1.5** | 0 | 0 | 0 |

**Step 3 :** We observe that – 1.5 is the least negative indicator and hence second column is the pivotal column. We examine the ratios of the quantities in the last column to that of the entries in the pivotal column.

$$160/2 = 80\ , \quad 240/2 = 120$$

The first one is the least, 2 is the pivot, first row is the pivotal row. *NZV* $s_1$ (*departing variable*) is replaced by $y$ . (*entering variable*)

**Step 4 :** We make the pivot 2 in the first row unity by multiplying first row by 1/2. The first simplex tableau is written.

| *NZV* | $x$ | $y$ | $s_1$ | $s_2$ | Qty |
|---|---|---|---|---|---|
| $y$ | 1/2 | **1** | 1/2 | 0 | 80 |
| $s_2$ | 3 | **2** | 0 | 1 | 240 |
| Δ | – 1 | **–1.5** | 0 | 0 | 0 |

Perform $R_2 \to -2R_1 + R_2$, $R_3 \to 1.5R_1 + R_3$

| NZV | $x$ | $y$ | $s_1$ | $s_2$ | Qty |
|---|---|---|---|---|---|
| $y$ | 1/2 | 1 | 1/2 | 0 | 80 |
| $s_2$ | 2 | 0 | −1 | 1 | 80 |
| Δ | −0.25 | 0 | 0.75 | 0 | 120 |

We observe that one of the indicator is negative and hence the process will continue.

Now the first column is the povital column. We find the ratios.

$$80 / (1/2) = 160, \quad 80 / 2 = 40$$

The second one is least. 2 is the pivot and $NZV$ $s_2$ is replaced by $x$. We make the pivot 2 in the second row unity by multiplying the second row by 1/2.

**The second simplex tableau is formed.**

| NZV | x | y | $s_1$ | $s_2$ | Qty |
|---|---|---|---|---|---|
| $y$ | 1/2 | 1 | 1/2 | 0 | 80 |
| $x$ | 1 | 0 | −1/2 | 1/2 | 40 |
| Δ | −0.25 | 0 | 0.75 | 0 | 120 |

Perform $R_1 \to -1/2 \cdot R_2 + R_1, \quad R_3 \to 0.25\, R_2 + R_3$

| NZV | $x$ | $y$ | $s_1$ | $s_2$ | Qty |
|---|---|---|---|---|---|
| $y$ | 0 | 1 | 3/4 | −1/4 | **60** |
| $x$ | 1 | 0 | −1/2 | 1/2 | **40** |
| Δ | 0 | 0 | 0.625 | 0.125 | **130** |

Since there are no negative indicators, the entires in the last column will give us the optimal solution.

Thus $(Z)_{Max}$ **= 130 at $x = 40$ and $y = 60$.**

---

32. *Maximize* $P = 2x + 3y + z$ *subject to the constraints,* $x + 3y + 2z \le 11$, $x + 2y + 5z \le 19$, $3x + y + 4z \le 25$, $x, y, z \ge 0$

>> Let us introduce slack variables $s_1, s_2, s_3$ to the inequalities to write them in the following form.

$$x + 3y + 2z + 1.s_1 + 0.s_2 + 0.s_3 = 11$$
$$x + 2y + 5z + 0.s_1 + 1.s_2 + 0.s_3 = 19$$
$$3x + y + 4z + 0.s_1 + 0.s_2 + 1.s_3 = 25$$
$$2x + 3y + z + 0.s_1 + 0.s_2 + 0.s_3 = P \text{ is the objective function.}$$

Solution by simplex method is presented in the following table

| NZV | $x$ | $y$ | $z$ | $s_1$ | $s_2$ | $s_3$ | Qty | Ratio | |
|---|---|---|---|---|---|---|---|---|---|
| $s_1$ | 1 | **3** | 2 | 1 | 0 | 0 | 11 | 11/3 = **3.33** | 3.33 is least, 3 is pivot, $s_1$ is replaced by $y$. Also $1/3 \cdot R_1$ |
| $s_2$ | 1 | **2** | 5 | 0 | 1 | 0 | 19 | 19/2 = 9.5 | |
| $s_3$ | 3 | 1 | 4 | 0 | 0 | 1 | 25 | 25/1 = 25 | |
| **Indicators** (Δ) | − 2 | **− 3** | − 1 | 0 | 0 | 0 | 0 | | |
| $y$ | 1/3 | **1** | 2/3 | 1/3 | 0 | 0 | 11/3 | | |
| $s_2$ | 1 | **2** | 5 | 0 | 1 | 0 | 19 | | $R_2 \to -2R_1 + R_2$ |
| $s_3$ | 3 | **1** | 4 | 0 | 0 | 1 | 25 | | $R_3 \to -R_1 + R_3$ |
| Δ | − 2 | **− 3** | − 1 | 0 | 0 | 0 | 0 | | $R_4 \to 3R_1 + R_4$ |
| $y$ | 1/3 | **1** | 2/3 | 1/3 | 0 | 0 | 11/3 | 11/3 ÷ 1/3 = 11 | 8 is least 8/3 is pivot $s_3$ is replaced by $x$ Also $3/8 \cdot R_3$ |
| $s_2$ | 1/3 | **0** | 11/3 | − 2/3 | 1 | 0 | 35/3 | 35/3 ÷ 1/3 = 35 | |
| $s_3$ | 8/3 | **0** | 10/3 | − 1/3 | 0 | 1 | 64/3 | 64/3 ÷ 8/3 = **8** | |
| Δ | **− 1** | **0** | 1 | 1 | 0 | 0 | 11 | | |
| $y$ | **1/3** | 1 | 2/3 | 1/3 | 0 | 0 | 11/3 | | $R_1 \to -1/3 \cdot R_3 + R_1$ |
| $s_2$ | **1/3** | 0 | 11/3 | − 2/3 | 1 | 0 | 35/3 | | $R_2 \to -1/3 \cdot R_3 + R_2$ |
| $x$ | **1** | 0 | 10/8 | − 1/8 | 0 | 3/8 | 8 | | |
| Δ | **− 1** | 0 | 1 | 1 | 0 | 0 | 11 | | $R_4 \to R_3 + R_4$ |
| $y$ | **0** | 1 | 1/4 | 3/8 | 0 | − 1/8 | **1** | | |
| $s_2$ | **0** | 0 | 13/4 | − 5/8 | 1 | − 1/8 | 9 | | |
| $x$ | **1** | 0 | 10/8 | − 1/8 | 0 | 3/8 | **8** | | |
| Δ | **0** | 0 | 9/4 | 7/8 | 0 | 3/8 | **19** | | **No negative indicators** |

Thus $(P)_{Max}$ **= 19 at** $x = 8$, $y = 1$, $z = 0$

---

**33.** *Use the simplex method to maximize* $Z = 3x + 4y$ *subject to the constraints* $2x + y \le 40$, $2x + 5y \le 180$, $x \ge 0$, $y \ge 0$.

>> Let us introduce slack variables $s_1$ and $s_2$ to the inequalities to write them in the following form.

$$2x + y + 1.s_1 + 0.s_2 = 40$$

$$2x + 5y + 0.s_1 + 1.s_2 = 180$$

$$3x + 4y + 0.s_1 + 0.s_2 = Z \text{ is the objective function.}$$

Solution by the simplex method is presented in the following table.

| NZV | $x$ | $y$ | $s_1$ | $s_2$ | Qty | Ratio | |
|---|---|---|---|---|---|---|---|
| $s_1$ | 2 | **1** | 1 | 0 | 40 | $40/1 = 40$ | 36 is least, 5 is the pivot. $s_2$ is replaced by $y$. Also $1/5 \cdot R_2$ |
| $s_2$ | 2 | **5** | 0 | 1 | 180 | $180/5 =$ **36** | |
| Indicators ( $\Delta$ ) | $-3$ | $\mathbf{-4}$ | 0 | 0 | 0 | | |
| $s_1$ | 2 | **1** | 1 | 0 | 40 | | $R_1 \to -R_2 + R_1$ |
| $y$ | 2/5 | **1** | 0 | 1/5 | 36 | | |
| $\Delta$ | $-3$ | $\mathbf{-4}$ | 0 | 0 | 0 | | $R_3 \to 4R_2 + R_3$ |
| $s_1$ | 8/5 | **0** | 1 | $-1/5$ | 4 | $4 \div 8/5 = 2.5$ | 2.5 is least, 8/5 is pivot, $s_1$ is replaced by $x$. Also $5/8 \cdot R_1$ |
| $y$ | 2/5 | **1** | 0 | 1/5 | 36 | $36 \div 2/5 = 90$ | |
| $\Delta$ | $\mathbf{-7/5}$ | 0 | 0 | 4/5 | 144 | | |
| $x$ | **1** | 0 | 5/8 | $-1/8$ | 5/2 | | |
| $y$ | **2/5** | 1 | 0 | 1/5 | 36 | | $R_2 \to -2/5 \cdot R_1 + R_2$ |
| $\Delta$ | $\mathbf{-7/5}$ | 0 | 0 | 4/5 | 144 | | $R_3 \to 7/5 \cdot R_1 + R_3$ |
| $x$ | **1** | 0 | 5/8 | $-1/8$ | **5/2** | | |
| $y$ | **0** | 1 | $-1/4$ | 1/4 | **35** | | |
| $\Delta$ | **0** | 0 | 7/8 | 5/8 | **147.5** | | **No negative indicators** |

**Thus $(Z)_{Max} = 147.5$ at $x = 2.5$ and $y = 35$**

---

**34.** *Use simplex method to maximize* $z = 2x + 4y$ *subject to the constraints* $3x + y \leq 22$, $2x + 3y \leq 24$, $x \geq 0$, $y \geq 0$.

>> Let us introduce slack variables $s_1$ and $s_2$ to the inequalities to write them in the following form.

$$3x + y + 1 \cdot s_1 + 0 \cdot s_2 = 22$$
$$2x + 3y + 0 \cdot s_1 + 1 \cdot s_2 = 24$$

$2x + 4y + 0 \cdot s_1 + 0 \cdot s_2 = z$ is the objective function.

Solution by the simplex method is presented in the following table.

| NZV | $x$ | $y$ | $s_1$ | $s_2$ | Qty | Ratio | |
|---|---|---|---|---|---|---|---|
| $s_1$ | 3 | 1 | 1 | 0 | 22 | $22/1 = 22$ | 8 is least, 3 is pivot. $s_2$ is replaced by $y$. |
| $s_2$ | 2 | 3 | 0 | 1 | 24 | $24/3 = 8$ | Also $1/3 \cdot R_2$ |
| **Indicators ($\Delta$)** | $-2$ | $-4$ | 0 | 0 | 0 | | |
| $s_1$ | 3 | 1 | 1 | 0 | 22 | | $R_1 \to -R_2 + R_1$ |
| $y$ | 2/3 | 1 | 0 | 1/3 | 8 | | |
| $\Delta$ | $-2$ | $-4$ | 0 | 0 | 0 | | $R_3 \to 4R_2 + R_3$ |
| $s_1$ | 7/3 | 0 | 1 | $-1/3$ | 14 | | |
| $y$ | 2/3 | 1 | 0 | 1/3 | 8 | | |
| $\Delta$ | 2/3 | 0 | 0 | 4/3 | 32 | | No negative indicators |

**Thus the maximum value of $Z$ is 32 at $x = 0$ and $y = 8$**

---

35. *Maximize* $P = 6x_1 + 2x_2 + 3x_3$ *subject to the constraints* $6x_1 + 5x_2 + 3x_3 \le 26$, $4x_1 + 2x_2 + 5x_3 \le 7$, $x_1 \ge 0$, $x_2 \ge 0$, $x_3 \ge 0$ *by applying simplex method.*

>> Let us introduce slack variables $s_1$ and $s_2$ to the inequalities to write them in the following form.

$$6x_1 + 5x_2 + 3x_3 + 1.s_1 + 0.s_2 = 26$$
$$4x_1 + 2x_2 + 5x_3 + 0.s_1 + 1.s_2 = 7$$

$6x_1 + 2x_2 + 3x_3 + 0.s_1 + 0.s_2 = P$ is the objective function.

Solution by the simplex method is presented in the following table.

| NZV | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | Qty | Ratio | |
|---|---|---|---|---|---|---|---|---|
| $s_1$ | 6 | 5 | 3 | 1 | 0 | 26 | 26/6 = 4.3 | 1.75 is least, 4 is pivot. $x_1$ replaces |
| $s_2$ | 4 | 2 | 5 | 0 | 1 | 7 | 7/4 = 1.75 | $s_2$. Also $1/4 \cdot R_2$ |
| **Indicators (Δ)** | −6 | −2 | −3 | 0 | 0 | 0 | | |
| $s_1$ | 6 | 5 | 3 | 1 | 0 | 26 | | $R_1 \to -6R_2 + R_1$ |
| $x_1$ | 1 | 1/2 | 5/4 | 0 | 1/4 | 7/4 | | |
| Δ | −6 | −2 | −3 | 0 | 0 | 0 | | $R_3 \to 6R_2 + R_3$ |
| $s_1$ | 0 | 2 | −9/2 | 1 | −3/2 | 31/2 | | |
| $x_1$ | 1 | 1/2 | 5/4 | 0 | 1/4 | **7/4** | | |
| Δ | 0 | 1 | 9/2 | 0 | 3/2 | **21/2** | | **No negative indicators** |

**Thus $(P)_{Max} = 21/2 = 10.5$ at $x_1 = 7/4, \ x_2 = 0, \ x_3 = 0$**

---

**36.** *Use the simplex method to maximize* $P = 4x_1 - 2x_2 - x_3$ *subject to the constraints* $x_1 + x_2 + x_3 \le 3, \ 2x_1 + 2x_2 + x_3 \le 4, \ x_1 - x_2 \ 0, \ x_1, \ x_2, \ x_3 \ge 0.$

>> Let us introduce slack variables $s_1, s_2, s_3$ to the inequalities to write them in the form

$$x_1 + x_2 + x_3 + 1.s_1 + 0.s_2 + 0.s_3 = 3$$
$$2x_1 + 2x_2 + x_3 + 0.s_1 + 1.s_2 + 0.s_3 = 4$$
$$x_1 - x_2 + 0.x_3 + 0.s_1 + 0.s_2 + 1.s_3 = 0$$

$4x_1 - 2x_2 - x_3 + 0.s_1 + 0.s_2 + 0.s_3 = P$ is the objective function.

Solution by the simplex method is presented in the following table.

| NZV | $x_1$ | $x_2$ | $x_3$ | $s_1$ | $s_2$ | $s_3$ | Qty | Ratio | |
|---|---|---|---|---|---|---|---|---|---|
| $s_1$ | 1 | 1 | 1 | 1 | 0 | 0 | 3 | $3/1 = 3$ | 0 is least.<br>1 is pivot.<br>$x_1$ replaces $s_3$ |
| $s_2$ | 2 | 2 | 1 | 0 | 1 | 0 | 4 | $4/2 = 2$ | |
| $s_3$ | 1 | $-1$ | 0 | 0 | 0 | 1 | 0 | $0/1 = 0$ | |
| $\Delta$ | $-4$ | 2 | 1 | 0 | 0 | 0 | 0 | | |
| $s_1$ | 1 | 1 | 1 | 1 | 0 | 0 | 3 | | $R_1 \to -R_3 + R_1$ |
| $s_2$ | 2 | 2 | 1 | 0 | 1 | 0 | 4 | | $R_2 \to -2R_3 + R_2$ |
| $x_1$ | 1 | $-1$ | 0 | 0 | 0 | 1 | 0 | | |
| $\Delta$ | $-4$ | 2 | 1 | 0 | 0 | 0 | 0 | | $R_4 \to 4R_3 + R_4$ |
| $s_1$ | 0 | 2 | 1 | 1 | 0 | $-1$ | 3 | $3/2 = 1.5$ | 1 is least<br>4 is pivot<br>$x_2$ replaces $s_2$<br>Also $1/4.R_2$ |
| $s_2$ | 0 | 4 | 1 | 0 | 1 | $-2$ | 4 | $4/4 = 1$ | |
| $x_1$ | 1 | $-1$ | 0 | 0 | 0 | 1 | 0 | $0/-1 = 0$ (Neglected) | |
| $\Delta$ | 0 | $-2$ | 1 | 0 | 0 | 4 | 0 | | |
| $s_1$ | 0 | 2 | 1 | 1 | 0 | $-1$ | 3 | | $R_1 \to -2R_2 + R_1$ |
| $x_2$ | 0 | 1 | $1/4$ | 0 | $1/4$ | $-1/2$ | 1 | | |
| $x_1$ | 1 | $-1$ | 0 | 0 | 0 | 1 | 0 | | $R_3 \to R_2 + R_3$ |
| $\Delta$ | 0 | $-2$ | 1 | 0 | 0 | 4 | 0 | | $R_4 \to 2R_2 + R_4$ |
| $s_1$ | 0 | 0 | $1/2$ | 1 | $-1/2$ | 0 | 1 | | |
| $x_2$ | 0 | 1 | $1/4$ | 0 | $1/4$ | $-1/2$ | 1 | | |
| $x_1$ | 1 | 0 | $1/4$ | 0 | $1/4$ | $1/2$ | 1 | | |
| $\Delta$ | 0 | 0 | $3/2$ | 0 | $1/2$ | 3 | 2 | | **No negative indicators** |

**Thus the maximum value of** $P$ is 2 at $x_1 = 1, x_2 = 1, x_3 = 0$

---

**37.** *Use simplex method to minimize* $P = x - 3y + 2z$ *subject to the constraints* $3x - y + 2z \le 7,\ -2x + 4y \le 12,\ -4x + 3y + 8z \le 10,\ x, y, z \ge 0.$

>> The given L.P.P is equivalent to maximizing the objective function $-P$ subject to the same constraints.

That is $-P = P' = -x + 3y - 2z$ is to be maximized.

Let us introduce slack variables $s_1, s_2, s_3$ to the inequalities to write them in the form

$$3x - y + 2z + 1.s_1 + 0.s_2 + 0.s_3 = 7$$
$$-2x + 4y + 0z + 0.s_1 + 1.s_2 + 0.s_3 = 12$$
$$-4x + 3y + 8z + 0.s_1 + 0.s_2 + 1.s_3 = 10$$
$$-x + 3y - 2z + 0.s_1 + 0.s_2 + 0.s_3 = P' \text{ is the objecive function.}$$

Solution by simplex method is presented in the following table.

| NZV | $x$ | $y$ | $z$ | $s_1$ | $s_2$ | $s_3$ | Qty | Ratio | |
|---|---|---|---|---|---|---|---|---|---|
| $s_1$ | 3 | **−1** | 2 | 1 | 0 | 0 | 7 | 7/−1 = −7 | 3 is least, 4 is pivot. |
| $s_2$ | −2 | **4** | 0 | 0 | 1 | 0 | 12 | 12/4 = 3 | $y$ replaces $s_2$. |
| $s_3$ | −4 | **3** | 8 | 0 | 0 | 1 | 10 | 10/3 = 3.3 | Also $1/4 \cdot R_2$ |
| **Indicators (Δ)** | 1 | **−3** | 2 | 0 | 0 | 0 | 0 | | |
| $s_1$ | 3 | **−1** | 2 | 1 | 0 | 0 | 7 | | $R_1 \to R_2 + R_1$ |
| $y$ | −1/2 | **1** | 0 | 0 | 1/4 | 0 | 3 | | |
| $s_3$ | −4 | **3** | 8 | 0 | 0 | 1 | 10 | | $R_3 \to -3R_2 + R_3$ |
| Δ | 1 | **−3** | 2 | 0 | 0 | 0 | 0 | | $R_4 \to 3R_2 + R_4$ |
| $s_1$ | **5/2** | **0** | 2 | 1 | 1/4 | 0 | 10 | 10 ÷ ( 5/2 ) = 4 | 4 is least 5/2 is pivot |
| $y$ | −1/2 | **1** | 0 | 0 | 1/4 | 0 | 3 | 3 ÷ −1/2 = −6 | $x$ replaces $s_1$ |
| $s_3$ | −5/2 | **0** | 8 | 0 | −3/4 | 1 | 1 | 1 ÷ −5/2 = −2/5 | Also ( 2/5 ).$R_1$ |
| Δ | **−1/2** | **0** | 2 | 0 | 3/4 | 0 | 9 | | |
| $x$ | **1** | 0 | 4/5 | 2/5 | 1/10 | 0 | 4 | | |
| $y$ | **−1/2** | 1 | 0 | 0 | 1/4 | 0 | 3 | | $R_2 \to 1/2 \cdot R_1 + R_2$ |
| $s_3$ | **−5/2** | 0 | 8 | 0 | −3/4 | 1 | 1 | | $R_3 \to 5/2 \cdot R_1 + R_3$ |
| Δ | **−1/2** | 0 | 2 | 0 | 3/4 | 0 | 9 | | $R_4 \to 1/2 \cdot R_1 + R_4$ |
| $x$ | **1** | 0 | 4/5 | 2/5 | 1/10 | 0 | 4 | | |
| $y$ | **0** | 1 | 2/5 | 1/5 | 3/10 | 0 | 5 | | |
| $s_3$ | **0** | 0 | 10 | 1 | −1/2 | 1 | 11 | | |
| Δ | 0 | 0 | 12/5 | 1/5 | 4/5 | 0 | 11 | | **No negative indicators** |

$\therefore \quad (P')_{Max} = 11$ at $x = 4$, $y = 5$, $z = 0$

**Thus the required minimum value of $P = -11$ at $x = 4, y = 5, z = 0$.**

**38.** *Solve by simplex method the following LPP: Maximize* $Z = 6x_1 + 9x_2$ *subject to* $2x_1 + 2x_2 \le 24$, $x_1 + 5x_2 \le 44$, $6x_1 + 2x_2 \le 60$ *and* $x_1, x_2 \ge 0$.

>> We prefer to divide the first and third constraints by 2 so that we have the constraints in the form $x_1 + x_2 \le 12$, $x_1 + 5x_2 \le 44$, $3x_1 + x_2 \le 30$.

Let us introduce slack variables $s_1$, $s_2$, $s_3$ to the inequalities to write them in the form,

$$x_1 + x_2 + 1.s_1 + 0.s_2 + 0.s_3 = 12$$
$$x_1 + 5x_2 + 0.s_1 + 1.s_2 + 0.s_3 = 44$$
$$3x_1 + x_2 + 0.s_1 + 0.s_2 + 1.s_3 = 30$$

$$6x_1 + 9x_2 + 0.s_1 + 0.s_2 + 0.s_3 = Z \quad \text{is the objective function.}$$

Solution by simplex method is presented in the following table.

| NZV | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | Qty | Ratio | |
|---|---|---|---|---|---|---|---|---|
| $s_1$ | 1 | 1 | 1 | 0 | 0 | 12 | $\frac{12}{1} = 12$ | 8.8 is least. 5 is pivot $x_2$ replaces $s_2$ Also $1/5 \cdot R_2$ |
| $s_2$ | 1 | 5 | 0 | 1 | 0 | 44 | $\frac{44}{5} = 8.8$ | |
| $s_3$ | 3 | 1 | 0 | 0 | 1 | 30 | $\frac{30}{1} = 30$ | |
| Δ | − 6 | − 9 | 0 | 0 | 0 | 0 | | |
| $s_1$ | 1 | 1 | 1 | 0 | 0 | 12 | | $R_1 \to -R_2 + R_1$ |
| $x_2$ | 1/5 | 1 | 0 | 1/5 | 0 | 44/5 | | |
| $s_3$ | 3 | 1 | 0 | 0 | 1 | 30 | | $R_3 \to -R_2 + R_3$ |
| Δ | − 6 | − 9 | 0 | 0 | 0 | 0 | | $R_4 \to 9R_2 + R_4$ |
| $s_1$ | 4/5 | 0 | 1 | − 1/5 | 0 | 16/5 | $\frac{16/5}{4/5} = 4$ | 4 is least. 4/5 is pivot $x_1$ replaces $s_1$ Also $5/4.R_1$ |
| $x_2$ | 1/5 | 1 | 0 | 1/5 | 0 | 44/5 | $\frac{44/5}{1/5} = 44$ | |
| $s_3$ | 14/5 | 0 | 0 | − 1/5 | 1 | 106/5 | $\frac{106/5}{14/5} = 7.6$ | |
| Δ | − 21/5 | 0 | 0 | 9/5 | 0 | 396/5 | | |

| NZV | $x_1$ | $x_2$ | $s_1$ | $s_2$ | $s_3$ | Qty | Ratio | |
|---|---|---|---|---|---|---|---|---|
| $x_1$ | [illegible] | 0 | 5/4 | −1/4 | 0 | 4 | | |
| $x_2$ | 1/5 | 1 | 0 | 1/5 | 0 | 44/5 | | $R_2 \to -1/5 \cdot R_1 + R_2$ |
| $s_3$ | 14/5 | 0 | 0 | −1/5 | 1 | 106/5 | | $R_3 \to -14/5 \cdot R_1 + R_3$ |
| Δ | −21/5 | 0 | 0 | 9/5 | 0 | 396/5 | | $R_4 \to 21/5 \cdot R_1 + R_4$ |
| $x_1$ | 1 | 0 | 5/4 | −1/4 | 0 | **4** | | |
| $x_2$ | 0 | 1 | −1/4 | 1/4 | 0 | **8** | | |
| $s_3$ | 0 | 0 | −7/2 | 1/2 | 1 | 10 | | |
| Δ | 0 | 0 | 21/4 | 3/4 | 0 | **96** | | **No negative indicators** |

**Thus the maximum value of $Z$ is 96 at $x_1 = 4$ and $x_2 = 8$**

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**39.** *Solve the following minimization problem by simplex method.*

*Objective function :* $P = -3x + 8y - 5z$

*Constraints :* $-x - 2z \leq 5$, $2x - 3y + z \leq 3$ $2x - 5y + 6z \leq 5$, $x_1$, $x_2$, $x_3 \geq 0$

>> Minimizing $P$ is equivalent to maximizing $-P = P' = 3x - 8y + 5z$

Let us introduce slack variables $s_1$, $s_2$, $s_3$ to the inequalities to write them in the following form

$$-x + 0.y - 2z + 1.s_1 + 0.s_2 + 0.s_3 = 5$$
$$2x - 3y + z + 0.s_1 + 1.s_2 + 0.s_3 = 3$$
$$2x - 5y + 6z + 0.s_1 + 0.s_2 + 1.s_3 = 5$$

$3x - 8y + 5z + 0.s_1 + 0.s_2 + 0.s_3 = P'$ is the objective function.

Solution by simplex method is presented in the following table.

| NZV | $x$ | $y$ | $z$ | $s_1$ | $s_2$ | $s_3$ | Qty | Ratio | |
|---|---|---|---|---|---|---|---|---|---|
| $s_1$ | – 1 | 0 | **– 2** | 1 | 0 | 0 | 5 | 5/– 2 = – 2.5 | 0.83 is least<br>6 is pivot |
| $s_2$ | 2 | – 3 | **1** | 0 | 1 | 0 | 3 | 3/1 = 3 | $z$ replaces $s_3$ |
| $s_3$ | 2 | – 5 | **6** | 0 | 0 | 1 | 5 | 5/6 = **0.83** | Also $1/6 \cdot R_3$ |
| Δ | – 3 | 8 | **– 5** | 0 | 0 | 0 | 0 | | |
| $s_1$ | – 1 | 0 | **– 2** | 1 | 0 | 0 | 5 | | $R_1 \to 2R_3 + R_1$ |
| $s_2$ | 2 | – 3 | **1** | 0 | 1 | 0 | 3 | | $R_2 \to -R_3 + R_2$ |
| $z$ | 1/3 | – 5/6 | **1** | 0 | 0 | 1/6 | 5/6 | | |
| Δ | – 3 | 8 | **– 5** | 0 | 0 | 0 | 0 | | $R_4 \to 5R_3 + R_4$ |
| $s_1$ | – 1/3 | – 5/3 | **0** | 1 | 0 | 1/3 | 20/3 | – *ve* | 1.3 is least.<br>5/3 is pivot |
| $s_2$ | 5/3 | – 13/6 | **0** | 0 | 1 | – 1/6 | 13/6 | 13/6 ÷ 5/3 = 1.3 | $x$ replaces $s_2$ |
| $z$ | 1/3 | – 5/6 | **1** | 0 | 0 | 1/6 | 5/6 | 5/6 ÷ 1/3 = 5/2 | Also $3/5 \cdot R_2$ |
| Δ | **– 4/3** | 23/6 | **0** | 0 | 0 | 5/6 | 25/6 | | |
| $s_1$ | **– 1/3** | – 5/3 | 0 | 1 | 0 | 1/3 | 20/3 | | $R_1 \to 1/3 \cdot R_2 + R_1$ |
| $x$ | **1** | – 13/10 | 0 | 0 | 3/5 | – 1/10 | 13/10 | | |
| $z$ | **1/3** | – 5/6 | 1 | 0 | 0 | 1/6 | 5/6 | | $R_3 \to -1/3 \cdot R_2 + R_3$ |
| Δ | **– 4/3** | 23/6 | 0 | 0 | 0 | 5/6 | 25/6 | | $R_4 \to 4/3 \cdot R_2 + R_4$ |
| $s_1$ | **0** | – 21/10 | 0 | 1 | 1/5 | 3/10 | 213/30 | | |
| $x$ | **1** | – 13/10 | 0 | 0 | 3/5 | – 1/10 | **13/10** | | |
| $z$ | **0** | – 2/5 | 1 | 0 | – 1/5 | 1/5 | **2/5** | | |
| Δ | **0** | 21/10 | 0 | 0 | 4/5 | 7/10 | **59/10** | | **No negative indications** |

$\therefore \quad (P')_{Max} = 59/10$ at $x = 13/10,\ y = 0,\ z = 2/5$

**Thus the minimum value of $P = -59/10$ at $x = 13/10, y = 0, z = 2/5$**

**40.** *Show that the L.P.P for minimizing the objective function* $Z = -2u - 3v - w$ *subject to the constraints* $u + 2v - 2w \leq 5$ , $3u - v - w \leq 2$ , $u$ , $v$ , $w \geq 0$ *has unbounded solution.*

>> Let $Z' = -Z = 2u + 3v + w$ for maximization be the objective function.

Let us introduce slack variables $s_1$ and $s_2$ to the inequalities to write them in the form

$$u + 2v - 2w + 1.s_1 + 0.s_2 = 5$$

$$3u - v - w + 0.s_1 + 1.s_2 = 2$$

$2u + 3v + w + 0.s_1 + 0.s_2 = Z'$ is the objective function.

Solution by simplex method is presented in the following table.

| NZV | $u$ | $v$ | $w$ | $s_1$ | $s_2$ | Qty | Ratio | |
|---|---|---|---|---|---|---|---|---|
| $s_1$ | 1 | 2 | −2 | 1 | 0 | 5 | 5/2 = **2.5** | 2.5 is least 2 is pivot. $v$ replaces $s_1$. Also $1/2 \cdot R_1$ |
| $s_2$ | 3 | [illegible] | −1 | 0 | 1 | 2 | Negative | |
| Δ | −2 | [illegible] | −1 | 0 | 0 | 0 | | |
| $v$ | 1/2 | 1 | −1 | 1/2 | 0 | 5/2 | | |
| $s_2$ | 3 | [illegible] | −1 | 0 | 1 | 2 | | $R_2 \to R_1 + R_2$ |
| Δ | −2 | −3 | −1 | 0 | 0 | 0 | | $R_3 \to 3R_1 + R_3$ |
| $v$ | 1/2 | **1** | −1 | 1/2 | 0 | 5/2 | $\frac{5/2}{-1} = -5/2$ | |
| $s_2$ | 7/2 | **0** | −2 | 1/2 | 1 | 9/2 | $\frac{9/2}{-2} = -9/4$ | |
| Δ | −1/2 | **0** | **−4** | 3/2 | 0 | 15/2 | | |

Since both the ratios are negative we cannot proceed further.

**Thus we conclude the L.P.P has unbounded solution.**

## EXERCISES

**Solve the following linear programming problems graphically.** [ **1-4** ]

1. Maximize $z = 2x + 4y$ subject to the constraints
   $3x + y \leq 22,\ 2x + 3y \leq 24\ ;\ x,\ y \geq 0$
2. Minimize $z = 2x + 3y$ subject to the constraints
   $x + y \geq 6,\ 2x + y \geq 7,\ x + 4y \geq 8,\ x,\ y \geq 0$
3. Maximize $z = 5x + 16y$ subject to the constraints
   $x + 6y \leq 18,\ x + 2y \leq 6\ ;\ x,\ y \geq 0$

4. Minimize $z = 60x + 50y$ subject to the constraints
$x + 2y \le 40,\ 3x + 2y \le 60\ ;\ x,\ y \ge 0$

**Solve the following L.P.P by applying simplex method [5-8]**

5. Maximize $P = 6x + 10y + 2z$ subject to the constraints
$2x + 4y + 3z \le 40,\ x + y \le 10,\ 2y + z \le 12\ ;\ x,\ y,\ z \ge 0$

6. Maximize $P = 3x + y + 2z$ subject to the constraints
$3x + 2y + z \le 11,\ 2x + 5y + z \le 19,\ x + 4y + 3z \le 25,\ x,\ y,\ z \ge 0$

7. Maximize $P = 5x + 3y$ subject to the constraints
$x + y \le 2,\ 5x + 2y \le 10,\ 3x + 8y \le 12\ ;\ x,\ y \ge 0$

8. Maximize $P = 7x + 12y + 16z$ subject to the constraints
$2x + y + z \le 1\ ;\ x + 2y + 4z \le 2,$

9. A radio factory produces two different types of transistors, ordinary model and special model. For greater efficiency the assembly and finishing process are performed in two different workshops. The ordinary model requires 3 hours of work in workshop-I and 4 hours in workshop II, while the special model required 6 hours in workshop - I and 4 hours in workshop II. Due to limited resources in skilled labour and materials only 180 hours of work can be done in workshop I and 200 hours of work in workshop-II. The factory makes a profit of Rs.30/- on each ordinary model and Rs.40/- on each special model. Assuming that it can sell all the transistors how many of each type should be produced in order to get maximum profit ? Solve this L.P.P by both graphical and simplex methods.

10. The owner of a dairy is trying to determine the correct blend of two types of feed. Both contain various percentages of four essential ingredients. Use the following data to determine the least cost blend.

| Ingredient | Percentage per kg of feed | | Minimum requirement in kg |
|---|---|---|---|
| | Feed 1 | Feed 2 | |
| 1 | 40 | 20 | 4 |
| 2 | 10 | 30 | 2 |
| 3 | 20 | 40 | 3 |
| 4 | 30 | 10 | 6 |
| Cost (Rs / kg) | 5 | 3 | |

## ANSWERS

1. $x = 0,\ y = 8\ ;\ z = 32$
2. $x = 5.3,\ y = 0.7\ ;\ z = 12.7$
3. $x = 0,\ y = 3\ ;\ z = 48$
4. $x = 10,\ y = 15\ ;\ z = 1350$
5. 84
6. 19
7. 10
8. 12
9. 40 ordinary and 10 special models. Maximum profit = Rs.1600/-.
10. $x_1 = 0.4kg$ (400 gms), $x_2 = 0$, Minimum cost Rs.2/-